

Dispersion equation for water waves with vorticity and Stokes waves on flows with counter-currents

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Abstract

The two-dimensional free-boundary problem of steady periodic waves with vorticity is considered for water of finite depth. We investigate how a flow with small-amplitude Stokes waves on the free surface bifurcates from a horizontal parallel shear flow in which counter-currents may be present. The bifurcation mechanism is described in terms of a dispersion equation; namely, wavelengths of Stokes waves bifurcate from the values defined by the roots of this equation. The latter generalizes that for irrotational waves and involves only quantities given on the horizontal free surface of the initial parallel shear flow. Sufficient conditions guaranteeing the existence of roots of the dispersion equation are obtained. Two particular vorticity distributions are considered in order to illustrate general results.

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1 Introduction

We study the two-dimensional nonlinear problem of steady waves in a horizontal open channel of uniform rectangular cross-section occupied by an inviscid, incompressible and heavy fluid, say, water. The water motion is assumed to be rotational which, according to observations, is the type of motion commonly occurring in nature (see, for example, [58, 59] and references cited therein). Our aim is to consider the bifurcation mechanism resulting in the formation of Stokes waves (periodic waves whose profiles rise and fall exactly once per wavelength) on a horizontal free surface of a parallel shear flow in which counter-currents may be present. A detailed study of such flows is given in our previous paper [50] under the assumptions that the flow velocity depends only on the vertical coordinate and it forms a basis for the present approach. Its crucial point is to determine a bifurcation wavelength for which purpose a dispersion equation is used. This technique is similar to the method that was earlier applied in our investigation of the irrotational Stokes waves (see [49], section 8.2.1). To the authors' best knowledge, the dispersion equation derived here (see (18) in section 1.4) was not used previously. In order to illustrate how the general results work for particular vorticity distributions several examples are presented.

1.1 Statement of the problem

Let a horizontal open channel of uniform rectangular cross-section be bounded below by the horizontal rigid bottom. Let water occupying the channel be bounded above by a free surface. The surface tension is neglected and the pressure is assumed to be constant on the free surface. The assumption that the water motion is two-dimensional and rotational and the incompressibility of

water allows us to seek the velocity field in the form $(\Psi_Y, -\Psi_X)$, where (X, Y) are appropriate Cartesian coordinates and Ψ is a *stream function* (see, for example, [52] and [55] for its definition). The vorticity distribution $\omega(\Psi)$ is supposed to be a prescribed $C_{loc}^{2,\alpha}$ -function on \mathbb{R} , $\alpha \in (0, 1)$, with bounded derivative. In the present paper, all variables are non-dimensional and chosen so that the constant volume rate of flow per unit span and the constant acceleration due to gravity are scaled to unity. For this purpose lengths and velocities are scaled to $(Q^2/g)^{1/3}$ and $(Qg)^{1/3}$, respectively; here Q and g are the dimensional quantities for the rate of flow and the gravity acceleration, respectively. We recall that $(Q^2/g)^{1/3}$ is the depth of the *critical uniform* stream in the irrotational case (see, for example, [4]).

Let (X, Y) be such that the X -axis lies in the longitudinal section of the canal's bottom and gravity acts in the negative Y -direction. Moreover, let the frame of reference be chosen so that the velocity field and the unknown free surface are time-independent in these coordinates. Assuming that $Y = \xi(X)$ represents the free-surface profile (here ξ is a positive, continuous function), we denote by \mathcal{D} the longitudinal section of the water domain, that is,

$$\mathcal{D} = \{-\infty < X < +\infty, 0 < Y < \xi(X)\}.$$

Since the surface tension is neglected, Ψ and ξ must satisfy the following free-boundary problem:

$$\Psi_{XX} + \Psi_{YY} + \omega(\Psi) = 0, \quad (X, Y) \in \mathcal{D}; \quad (1)$$

$$\Psi(X, 0) = 0, \quad X \in \mathbb{R}; \quad (2)$$

$$\Psi(X, \xi(X)) = 1, \quad X \in \mathbb{R}; \quad (3)$$

$$|\nabla_{X,Y} \Psi(X, \xi(X))|^2 + 2\xi(X) = 3r, \quad X \in \mathbb{R}. \quad (4)$$

In the last relation (Bernoulli's equation), $r > 0$ is the problem's parameter referred to as the non-dimensional total head (also known as Bernoulli's constant). This formulation has long been known (see, for example, [32]); its derivation from the governing equations and the assumptions about the boundary behaviour of water particles can be found in [16].

Considering the question of formation of Stokes waves on a flat free surface of a shear flow, we suppose that $\Psi \in C^{2,\alpha}(\bar{\mathcal{D}})$ and $\xi \in C^{2,\alpha}$, $0 < \alpha < 1$.

1.2 Background

Nonlinear theory of gravity water waves with vorticity has a long history which dates back to the 1800s, when Gerstner [33] found his remarkable explicit solution (see [6] for a modern approach to this solution; in [36] some its properties are obtained). Notice that Scott Russell and Stokes published their pioneering works about irrotational water waves only in the 1840s (see [54] and [56], respectively). Nevertheless, rotational waves have been studied to a less extent than irrotational ones (see the recent survey paper [57], where an extensive bibliography is provided). Only a few articles treating waves with vorticity rigorously were published during the 20th century, and at least three of them are of lasting interest. As early as 1934, Dubreil-Jacotin [32] had proved the first existence result for these waves (in the 1950s, her work was extended by Goyon [34]). For this purpose she introduced a partial hodograph transform which is a very convenient tool to investigate rotational waves provided the horizontal component of the relative velocity in the flow does not change sign. In 1978, Keady and Norbury [45] obtained bounds on the total head of flow and free-surface profiles, but their results were proved only for a rather small class of vorticity distributions. In the authors' work [51], these bounds were generalized to the case of arbitrary Lipschitz distributions.

During the past decade, a substantial body of rigorous results about waves with vorticity has appeared, the first of which was the article [16] by Constantin and Strauss. They used the partial

hodograph transform for obtaining a global branch of large-amplitude, Stokes waves. No counter-currents are allowed in this work as well as in the most of other papers that are briefly characterized by placing them into the following overlapping groups each of which covers a particular topic.

- The existence of waves is shown through the local/global bifurcation mechanism in [5, 19, 20, 35, 37, 39, 41, 62, 63, 65, 66, 67].
- Regularity (in particular, analyticity) of the stream function and/or streamlines is investigated in [13, 20, 31, 42].
- Symmetry of periodic and solitary waves is studied in [10, 11, 12, 25, 38, 40].
- Stability and instability of waves is considered in [18] and [43], respectively.
- The works [7, 15, 17, 19, 23, 24, 26, 27, 30, 44, 61, 62, 63, 64, 67] deal with some other properties of waves.
- Deep-water waves are treated in [11, 25, 37, 41].
- Solitary waves are investigated in [35, 39, 40].
- The papers [8, 9, 14, 28, 35] are devoted to various forms of a dynamical system approach to waves with vorticity.
- Numerical results about rotational waves are presented, for example, in [46, 47].

Now we turn to a few articles [20, 28, 29, 30, 68], in which rotational waves on flows with counter-currents are studied (the last of these papers contains an extensive list of references and a brief review of the literature on water waves with vorticity). The works [20, 30, 68] are concerned with the case of a constant vorticity, whereas a linear vorticity is considered in [28, 29]. Since the results obtained in the cited papers are closely related to ours, we discuss them in greater detail.

It should be emphasized that counter-currents are not considered as the principal point of those articles. Instead, the authors concentrate their attention on the appearance of internal *stagnation points* and the so-called *critical layers*. A critical layer is a connected subset of the water domain consisting of closed streamlines and stagnation points. See, for example, figures 1 and 3 in [68] and figure 3 in [20]. However, there exist degenerate cases like that shown in figure 2, [68]. From a geometric point of view critical layer is a horizontal array of cat's eye patterns anticipated by Kelvin [60] (see also [22], p. 141). Therefore, such a layer, generally speaking, separates two other layers with opposite directions of flow.

First, the existence of steady waves on a flow of constant vorticity in which a critical layer (and so, a counter-current) is present was established by Ehrnström and Villari [30], who studied streamlines and particle paths in the framework of linear theory. It was Wahlén [68], who proved the existence of small-amplitude Stokes waves with a constant vorticity and discovered that in the reference frame moving with the wave there is a critical layer. (As is discussed in [68], the term itself has a long history, but was mainly used in the framework of mathematical models other than rotational water waves.) Recently, Constantin and Varvaruca [20] proposed an approach to Stokes waves with a constant vorticity that differs from that in [68] and is based on the conformal mapping technique that imposes no restriction on the geometry of the free surface profile which, in particular, can be overhanging. This technique, as the authors claim, opens up the way to using global bifurcation theory. Besides, the physical relevance of the problem with a constant vorticity is discussed in [20].

In the articles [28, 29], Ehrnström *et al.* studied the case of linear vorticity when multiple counter-currents and critical layers exist. In [29], the results obtained in [68] were extended, yielding the existence of small-amplitude waves with arbitrarily many critical layers (including the so-called bichromatic waves). Qualitative and some quantitative properties of waves the existence of which is proved in [29] are investigated in [28]. In particular, a classification of vorticity distributions is proposed and the bifurcation relations are given for different cases.

1.3 Stream solutions and their basic properties

A pair $(u(Y), h)$, where $h = \text{const}$, is called a *stream solution* of problem (1)–(4) when

$$\Psi(X, Y) = u(Y) \quad \text{and} \quad \xi(X) = h$$

satisfy this problem. Such solutions are studied in detail in [50], where they are also discussed in terms of the unified theory of conjugate flows developed by Benjamin [3]; here we restrict ourselves to a summary of those results.

Seeking stream solutions, we write problem (1)–(4) in the form:

$$u'' + \omega(u) = 0, \quad u(0) = 0, \quad u(h) = 1, \quad |u'(h)|^2 + 2h = 3r, \quad (5)$$

where $u' = u_Y$. By $U(Y; s)$, we denote a unique solution existing on the whole \mathbb{R} for the first equation (5) complemented by the following Cauchy data:

$$U(0; s) = 0, \quad U'(0; s) = s, \quad \text{where } s \in \mathbb{R}. \quad (6)$$

It is reasonable to use s for parametrising solutions, and it occurs that all stream solutions are parameterised by

$$s \geq s_0 = \sqrt{2 \max_{0 \leq \tau \leq 1} \Omega(\tau)}, \quad \text{where } \Omega(\tau) = \int_0^\tau \omega(t) dt. \quad (7)$$

Integrating the first equation (5) for $U(Y; s)$, one obtains

$$(U')^2 + 2\Omega(U) = s^2. \quad (8)$$

Moreover, we get that if $h(s) > 0$ is found for some s so that

$$U(h(s); s) = 1 \quad \text{and} \quad s^2 - 2\Omega(U(h(s); s)) + 2h(s) = 3r \quad (9)$$

hold simultaneously, then the pair $(U(Y; s), h(s))$ is a stream solution (note that in the second relation (9), the expression from (8) is substituted). These solutions exist only for $r \geq r_c$, where r_c is defined by formula (15) below, and the set of stream solutions corresponding to a particular r can be obtained by virtue of the following procedure (see details in [50]).

One takes some $s \geq s_0$ [see (7)] and finds $U(Y; s)$ from the Cauchy problem described above. In fact, s depends on r and we shall describe below how to get s for a given r (note that several values of s might correspond to some particular r). Then one obtains the values of depth for which purpose the first relation (9) serves. Let us outline the corresponding scheme involving two auxiliary quantities which we denote $\tau_\pm(s)$ and $y_\pm(s)$ (see [50], section 3, for their properties).

If the equation $2\Omega(\tau) = s^2$ with $s \geq s_0$ has finite positive and negative roots, then by $\tau_+(s)$ [$\tau_-(s)$] we denote the smallest positive root [the largest negative root, respectively]. If there is no finite positive [negative] root, then we put $\tau_+(s) = +\infty$ [$\tau_-(s) = -\infty$, respectively]. Notice that $\tau_+(s) \geq 1$ if and only if $s \geq s_0$. Now we set

$$y_\pm(s) = \int_0^{\tau_\pm(s)} \frac{d\tau}{\sqrt{s^2 - 2\Omega(\tau)}} \quad \text{for } s \geq s_0.$$

The introduced quantities are such that $(y_-(s), y_+(s))$ is the maximal interval, where $U(Y; s)$ increases strictly monotonically, and $\tau_+(s)$ [$\tau_-(s)$] is the supremum [infimum, respectively] of $U(Y; s)$ on this interval. More precisely, $U(Y; s)$ has the following properties.

- If $y_+(s) = +\infty$ and $y_-(s) = -\infty$, then $U(Y; s)$ increases strictly monotonically for all $Y \in \mathbb{R}$.

- If $y_-(s) = -\infty$ and $y_+(s) < +\infty$, then $U(Y; s)$ is bimonotonic and attains its maximum $\tau_+(s)$ at $Y = y_+(s)$.
- If $y_-(s) > -\infty$ and $y_+(s) = +\infty$, then $U(Y; s)$ is bimonotonic and attains its minimum $\tau_-(s)$ at $Y = y_-(s)$.
- If both $y_+(s)$ and $y_-(s)$ are finite, then $U(Y; s)$ is harmonic-like; it attains one of its minima at $Y = y_-(s)$ and one of its maxima at $Y = y_+(s)$. Moreover, $U(Y; s)$ increases strictly monotonically from $\tau_-(s)$ to $\tau_+(s)$ on $[y_-(s), y_+(s)]$.

Furthermore, it is proved in [50], section 4.2, that for $s \geq s_0$

$$h(s) = \int_0^1 \frac{d\tau}{\sqrt{s^2 - 2\Omega(\tau)}} \quad (10)$$

is the smallest positive value of the depth such that $U(h(s); s) = 1$. By $h_j^{(+)}(s)$, $j = 0, 1, \dots$, we denote the whole sequence (possibly finite) of depth values such that $U(h_j^{(+)}(s); s) = 1$ for a given s , and we have

$$h_{2k}^{(+)}(s) = h(s) + 2k[y_+(s) - y_-(s)], \quad j = 2k; \quad (11)$$

$$h_{2k+1}^{(+)}(s) = h(s) + 2[y_+(s) - h(s)] + 2k[y_+(s) - y_-(s)], \quad j = 2k + 1. \quad (12)$$

Here, $k = 0, 1, \dots$, and so $h(s)$ [see formula (10)] is included into (11), where it appears as $h_0^{(+)}(s)$. If both $y_+(s)$ and $y_-(s)$ are finite, then formulae (11) and (12) give finite values for all $k = 0, 1, \dots$. Otherwise the first of them gives a finite value only for $k = 0$. Formula (12) also gives a finite value for $k = 0$ provided $y_+(s)$ is finite. Note that both formulae coincide when $y_+(s) = h(s)$ that is equivalent to the equality $\tau_+(s) = 1$.

For $s > s_0$, there exists the following sequence of solutions

$$h_j^{(-)}(s) = h_j^{(+)}(s) - 2y_-(s), \quad j = 0, 1, \dots, \quad (13)$$

of the equation $U(h(s); -s) = 1$, and so this sequence also gives flow depths. The number of finite elements in the sequence (13) depends on whether $y_+(s)$ and $y_-(s)$ are finite or not. It is clear that the value $h_j^{(-)}(s)$ is infinite when either $h_j^{(+)}(s) = +\infty$ or $y_-(s) = -\infty$, whereas $h_j^{(-)}(s)$ is finite otherwise.

The subscript (superscript) at $h_j^{(\pm)}(s)$ points at the number of counter-currents (the direction of flow in the near-bottom layer, respectively) in the following way. If the depth is equal to $h_j^{(+)}$, then the number of layers with alternating directions of flow is equal to $j + 1$, whereas the direction of flow in the near-bottom layer coincides with the direction of the so-called critical unidirectional flow, which always exists, corresponds to $r = r_c$ and has the minimum depth among all possible flows. If the depth is equal to $h_j^{(-)}$, then there are $j + 2$ layers with alternating directions of flow and the near-bottom layer has the direction opposite to that of the critical flow.

Let us summarize our algorithm for finding stream solutions. We find appropriate values of s from the second relation (9) (Bernoulli's equation):

$$\mathcal{R}_j^{(\pm)}(s) = r, \quad \text{where } \mathcal{R}_j^{(\pm)}(s) = \frac{1}{3} \left[s^2 - 2\Omega(1) + 2h_j^{(\pm)}(s) \right], \quad j = 0, 1, \dots, \quad (14)$$

and the last expression is obtained with by virtue of (8). Then the stream solution

$$\left(U(Y; s), h_j^{(+)}(s) \right) \quad \left[\left(U(Y; -s), h_j^{(-)}(s) \right) \right]$$

corresponds to the root s of equation (14) with the subscript j and the superscript $+$ $[-]$.

Let us turn to the definition of r_c . Since for every $s > s_0$ the sequences $\mathcal{R}_j^{(+)}(s)$ and $\mathcal{R}_j^{(-)}(s)$ are increasing (they might be infinite or finite depending on the vorticity distribution), we have that the graph of $\mathcal{R}_0^{(+)}(s)$ (it is convex as is shown in [50], section 5.1) lies below those corresponding to other functions $\mathcal{R}_j^{(\pm)}(s)$. Therefore, we put

$$r_c = \min_{s \geq s_0} \mathcal{R}_0^{(+)}(s) \quad (15)$$

and this value is always attained at some $s_c > s_0$. Moreover, for all $r > r_c$ some equations in the sequence (14) have roots; at least two such roots exist for every r . It was said above that for each s obtained from the sequence of equations (14) one finds $U(Y; s)$ and $h(s)$. Thus, the set of all stream solutions corresponding to $r \geq r_c$ is determined.

Finally, it must be mentioned that if

$$r_0 = \lim_{s \rightarrow s_0 + 0} \mathcal{R}_0^{(+)}(s) \quad (16)$$

is finite, then among the flows corresponding to every $r > r_0$ there exists at least one flow that has a counter-current. These flows are defined by stream solutions obtained above and flows with several counter-currents can exist among them.

1.4 Main results

The existence theorem for Stokes waves involves three assumptions. Our first assumption concerns determining a shear flow from which Stokes waves bifurcate, namely, we suppose the following.

(I) The inequality $r > r_c$ holds. As was said above, it guarantees that some equations in the sequence (14) have roots. Moreover, at least two roots exist for every $r > r_c$, and we choose those which are greater than s_0 . Let s_* be one of them, and so we have that $U'(h; s_*) \neq 0$ (see [50]).

The next assumption is related to the so-called dispersion equation and properties of its roots. Similarly to the case of irrotational waves every of these roots determines a bifurcation wavelength as a function of r and h (of course, h itself depends on r). In the case of waves with vorticity, more than one bifurcation wavelength can exist for a given choice of a stream solution and this depends on the vorticity distribution (see section 5). On the contrary, only one wavelength exists for a given admissible r in the irrotational case because the corresponding dispersion equation can be written as follows (see [49], p. 478):

$$\tau \coth h\tau = h^2. \quad (17)$$

Here $h > 1$ is defined by r and its value is equal to the non-dimensional depth of the subcritical uniform stream (for convenience, the depth of the critical stream is scaled to unity). It is clear that (17) has only one positive root which is equal to the wavenumber $2\pi/\Lambda_0$ corresponding to the bifurcation wavelength. It is worth mentioning that the problem about periodic irrotational waves has another statement that leads to a different form of the dispersion equation. The latter involves the phase velocity of waves (see, for example, formula (1) in [69], ch. 71); however, this form is less convenient for generalising to the case of waves with vorticity.

In the rotational case, the stream solution $(U(\cdot; s_*), h(s_*))$ corresponding to the chosen s_* yields the following dispersion equation (see the derivation in section 2.4):

$$\sigma(\tau) = 0, \quad \text{where } \sigma(\tau) = \kappa \gamma'(h, \tau) - \kappa^{-1} + \omega(1). \quad (18)$$

In the last function, $\kappa = U'(h)$, whereas $\gamma(Y, \tau)$ solves the following problem:

$$-\gamma'' + [\tau^2 - \omega'(U)]\gamma = 0, \quad \gamma(0, \tau) = 0, \quad \gamma(h, \tau) = 1, \quad (19)$$

in which h depends on r through the root s_* of Bernoulli's equation. Note also that

$$\kappa = \pm\sqrt{3r-2h} \quad \text{when } \pm U'(h) > 0.$$

If τ^2 is not a Dirichlet eigenvalue of the operator $d^2/dY^2 + \omega'(U)$ (there might exist a finite number of such eigenvalues), then problem (19) has a unique solution. Therefore, $\gamma(Y, \tau)$ and $\sigma(\tau)$ are defined for all these values of τ and are smooth and even functions of τ . There are vorticity distributions for which the Dirichlet spectrum is not empty, and so $\gamma(Y, \tau)$ and $\sigma(\tau)$ are not defined for some values of τ . For example, this is true when $\omega(\tau)$ is a linear function positive for $\tau > 0$ (see section 5.2).

Let us list some properties of equation (18). First, the function $\sigma(\tau)$ is well-defined only when $s_* \neq s_0$ because otherwise κ vanishes. (Note that the equality $s_* = s_0$, excluded by assumption (I), holds only when r_0 defined by formula (16) is finite.) Second, if ω vanishes identically, then (18) coincides with (17) because $\kappa = h^{-1}$ and $U = Y/h$ in this case. Then the solution of (19) is $\gamma = \sinh Y\tau / \sinh h\tau$ which gives (17). Third, if zero is not a Dirichlet eigenvalue of $d^2/dY^2 + \omega'(U)$ (this is equivalent to the inequality $(dh/ds)(s_*) \neq 0$ and follows from assumption (III) formulated below), then the following equality holds (see formula (55) in Proposition 3.3):

$$\sigma(0) = -\frac{2}{3\kappa} \left[\frac{d\mathcal{R}}{ds}(s_*) \middle/ \frac{dh}{ds}(s_*) \right].$$

For the sake of brevity, here and below \mathcal{R} stands for the function $\mathcal{R}_j^{(\pm)}$ that defines the stream solution (U, h) which is involved in (18) and (19). The last equality is used for investigating whether equation (18) has roots. In particular, the existence of a positive root in the absence of the above mentioned Dirichlet eigenvalues follows from positivity of the fraction in the square brackets. On the other hand, if there is a Dirichlet eigenvalue, then (18) has at least one root greater than this eigenvalue [see Proposition 3.3 (ii)]. Moreover, if the fraction in the square brackets is positive, then there also exists a root between zero and this eigenvalue. Finally, for any $r > r_c$ the equation $\mathcal{R}_0^{(+)}(s) = r$ has a root $s_* > s_c$. If the stream solution (U, h) corresponds to this s_* and σ defined by (U, h) is a continuous function, then equation (18) has no positive solutions. The last fact has the well-known analogue for zero vorticity: only solitary waves exist in the supercritical case (see, for example, [2] and [48]).

Now we turn to the second assumption.

(II) The dispersion equation (18) has at least one positive root, say, τ_0 such that: (a) $\sigma'(\tau_0) \neq 0$, which means that this root is simple; (b) none of the values τ_0/k ($k = 1, 2, \dots$) is a root of (18).

In section 3, we consider conditions guaranteeing that this assumption is fulfilled. Now we are in a position to formulate the following.

Definition of a bifurcation wavelength. Let assumptions (I) and (II) hold, and let τ_0 be a root of (18) existing by assumption (II). Then we put $\Lambda_0 = 2\pi/\tau_0$, and so Λ_0 is a wavelength of linear rotational waves on the shear flow described by the stream solution (U, h) corresponding to s_* introduced after assumption (I).

If equation (18) has more than one root satisfying conditions (a) and (b) of assumption (II), then each of these roots defines a wavelength of linear waves with vorticity and all these waves exist on the free surface of one and the same shear flow. Our last assumption is as follows.

(III) If the boundary value problem

$$\varphi_{XX} + \varphi_{YY} + \omega'(U)\varphi = 0 \quad \text{in } \mathbb{R} \times (0, h), \quad \varphi(X, 0) = 0, \quad \varphi(X, h) = 0,$$

has a weak solution such that it is Λ_0 -periodic in X , then this solution is trivial.

In particular, this implies that zero is not a Dirichlet eigenvalue of the operator $d^2/dY^2 + \omega'(U)$ considered on $(0, h)$, and so the function $\sigma(\tau)$ is defined for τ belonging to a neighbourhood of zero.

Let us describe function spaces that are used in what follows. For a non-negative integer k and $\alpha \in (0, 1)$ by $\Pi_{\Lambda_0}^{k, \alpha}$ we denote the space of Λ_0 -periodic, even $C^{k, \alpha}$ -functions on \mathbb{R} (cf. [49], p. 478). Furthermore, let the strip \mathcal{D} be a $C^{k, \alpha}$ -domain whose boundary is Λ_0 -periodic and even in X . Then $\Pi_{\Lambda}^{k, \alpha}(\bar{\mathcal{D}})$ is the space of $C^{k, \alpha}$ -functions on $\bar{\mathcal{D}}$ which are Λ -periodic and even in X . Now we are in a position to formulate the following result.

Main Theorem. *Let $\omega \in C^{2, \alpha}(\mathbb{R})$, $\alpha \in (0, 1)$, and let $r > 0$ satisfy assumption (I) and be such that assumptions (II)–(III) hold for a stream solution (U, h) defined by r . Then there exist $\varepsilon > 0$ and a continuous mapping $t \mapsto (\lambda(t), \zeta(\cdot, t))$ from $\{t \in \mathbb{R} : |t| \in (0, \varepsilon)\}$ to a neighbourhood of $(0, 0)$ in $\mathbb{R} \times \Pi_{\Lambda_0}^{2, \alpha}$, where $\Lambda_0 = 2\pi/\tau_0$ and τ_0 is a positive root of equation (18) existing by assumption (II). Moreover, the components of this mapping have the following properties.*

(i) *The first component $\lambda(t) \not\equiv 0$ and $\lambda(t) \rightarrow 0$ as $|t| \rightarrow 0$, whereas $\zeta(x, t) \not\equiv 0$ for every t and $x \in [-\Lambda_0/2, \Lambda_0/2]$, and the asymptotic formula holds*

$$\zeta(x, t) = t \cos \frac{2\pi x}{\Lambda_0} + \zeta_*(x, t), \quad (20)$$

where $\|\zeta_*(\cdot, t)\|_{\Pi_{\Lambda_0}^{2, \alpha}} = o(t)$ as $|t| \rightarrow 0$.

(ii) *If the upper boundary of \mathcal{D} has the form*

$$\xi(X, t) = \frac{h}{1 + \lambda(t)} + \zeta \left(\frac{X}{1 + \lambda(t)}, t \right), \quad (21)$$

that is, $\xi(X, t)$ a Λ -periodic function of X with $\Lambda = \Lambda_0[1 + \lambda(t)]$, then problem (1)–(3) has a solution $\Psi(\cdot, t) \in \Pi_{\Lambda}^{2, \alpha}(\bar{\mathcal{D}})$ such that the pair $(\Psi(X, Y, t), \xi(X, t))$ satisfies condition (4). Moreover,

$$\Psi(X, Y, t) = U \left(Y \frac{h}{\xi(X, t)} \right) + \Psi_*(X, Y, t), \quad (22)$$

where $\|\Psi_*(\cdot, t)\|_{\Pi_{\Lambda}^{2, \alpha}(\bar{\mathcal{D}})} = O(t)$ as $t \rightarrow 0$.

This theorem means that (Ψ, ξ) is a Λ -periodic Stokes-wave solution of problem (1)–(4); formulae (20) and (22) give the asymptotic representations for its components. Thus, this solution is a perturbation of the stream solution (U, h) . Relations (20) and (21) immediately yield this for ξ , whereas for Ψ this fact will be shown in the theorem's proof, for which purpose the form of function Ψ_* will be specified (see section 4.2). Moreover, (Ψ, ξ) belongs to a family of solutions describing Stokes waves with wavelengths close to Λ_0 provided the latter is defined by a simple root of the dispersion equation (18). Note that if r_0 is finite and $r > r_0$, then (U, h) describes a shear flow with counter-currents (see [50] and section 5, where examples are considered).

In conclusion of this section, it is worth emphasizing that the mechanism of wave bifurcation is substantially more complicated for waves with vorticity comparing with that for irrotational waves. First, for every $r > r_c$ there are exactly two uniform flows (sub- and supercritical) in the irrotational case, whereas there can be as many rotational flows of constant depth as one pleases which, for example, is the case for linear positive vorticity (see section 5.2 below). Second, for a given depth of the subcritical irrotational uniform flow the dispersion equation (17) defines only a single value of the bifurcation wavelength. On the other hand, the structure of roots of the rotational dispersion equation (18) depends on the value of depth (the second component of the stream solution), which, in its turn, depends on $r > r_c$. This results in the fact that the number of values of the bifurcation wavelength can be as large as one pleases, which again is the case for linear positive vorticity.

2 Operator form of problem (1)–(4); dispersion equation

We prove Main Theorem using the Crandall–Rabinowitz theorem (see Theorem 4.1 in section 4). In order to apply it to problem (1)–(4), the latter must be transformed into a nonlinear operator equation.

2.1 Reformulation of the problem

First, we have to choose a horizontal shear flow from which Stokes waves can bifurcate. According to assumption (I), we fix $r > r_c$, where the critical value r_c is defined by formula (15) and depends on the vorticity distribution. Then equation (14) has a solution s_* at least for one function $\mathcal{R}_j^{(\pm)}$, $j = 0, 1, \dots$, and the same assumption implies that $s_* \neq s_0$. The stream solution corresponding to s_* is as follows:

$$(U_j^{(\pm)}(Y; s_*), h_j^{(\pm)}(s_*)). \quad (23)$$

Here the second component is given by one of the formulae (11)–(13) and expressions for the first component can be found in [50], section 3, whereas its properties are described above in section 1.3. For the sake of brevity, this solution will be denoted (U, h) in what follows, whereas \mathcal{R} will stand for the function $\mathcal{R}_j^{(\pm)}$ defining (U, h) . Note that $U'(h, s_*) \neq 0$, which is also a consequence of assumption (I). If for some $r > r_c$ equation (14) has roots for several functions $\mathcal{R}_j^{(\pm)}$, then each of these roots can be used for obtaining a dispersion equation through the corresponding solution (23). If assumption (II) is fulfilled for roots of several dispersion equations, then more than one bifurcation wavelength corresponds to the chosen r .

Now we reformulate the problem in order to introduce a small parameter so that the Crandall–Rabinowitz theorem is applicable to a single operator equation resulting from the amended statement. Notice that the chosen stream solution (U, h) defines the wavelength $\Lambda_0 > 0$ through the dispersion equation (18), because (U, h) is taken so that assumption (II) holds for (18) and its roots. Let Λ denote the wavelength of the sought Stokes waves that perturb the free surface $Y = h$ of the shear flow. Assuming that Λ is close to Λ_0 , we put $\Lambda = \Lambda_0(1 + \lambda)$, and so λ is a small parameter to be found along with the stream function and the wave profile.

For transforming problem (1)–(4) into a new one depending on λ , we introduce the following variables:

$$x = \frac{X\Lambda_0}{\Lambda}, \quad y = \frac{Y\Lambda_0}{\Lambda}, \quad \eta(x) = \frac{\xi(X)\Lambda_0}{\Lambda}, \quad \psi(x, y) = \Psi(X, Y). \quad (24)$$

Then \mathcal{D} is changed to the curved strip $D = \{-\infty < x < +\infty, 0 < y < \eta(x)\}$, and the problem takes the form:

$$\psi_{xx} + \psi_{yy} + (1 + \lambda)^2 \omega(\psi) = 0, \quad (x, y) \in D; \quad (25)$$

$$\psi(x, 0) = 0, \quad x \in \mathbb{R}; \quad (26)$$

$$\psi(x, \eta(x)) = 1, \quad x \in \mathbb{R}; \quad (27)$$

$$|\nabla_{x,y} \psi(x, y)|^2 + 2(1 + \lambda)^3 y = 3r(1 + \lambda)^2, \quad y = \eta(x), \quad x \in \mathbb{R}. \quad (28)$$

Now we seek (ψ, η) and λ , so that η is a non-constant, Λ_0 -periodic, even function, whereas ψ is Λ_0 -periodic and even in x .

In view of the boundary condition (27), equation (28) takes the form:

$$\frac{\partial \psi}{\partial n}(x, \eta(x)) - (1 + \lambda)[3r - 2(1 + \lambda)\eta(x)]^{1/2} = 0, \quad x \in \mathbb{R}, \quad (29)$$

provided $U'(h) > 0$; here n is the unit normal directed outwards of D . If $U'(h) < 0$, then we have

$$\frac{\partial \psi}{\partial n}(x, \eta(x)) + (1 + \lambda)[3r - 2(1 + \lambda)\eta(x)]^{1/2} = 0, \quad x \in \mathbb{R} \quad (30)$$

instead of (29). In what follows, we restrict our considerations to (29), just formulating the results that concern (30).

2.2 Reduction to a single operator equation

As in the case of irrotational waves (see [49], section 3), it is convenient to reduce problem (25)–(28) to a single equation. The first step is to transform the curved strip D into $S = \mathbb{R} \times (0, h)$. For this purpose we change the vertical coordinate y to the following one:

$$z = y \frac{h}{\eta(x)} \quad \left[\text{or } z = Y \frac{h}{\xi(X)} \text{ according to (24)} \right]. \quad (31)$$

Setting $u(z) = U(Y)$, we define new unknown functions $\zeta(x)$ on \mathbb{R} and $\phi(x, z)$ on \bar{S} as follows:

$$\zeta(x) = \eta(x) - \frac{h}{1 + \lambda} \quad \text{and} \quad \phi(x, z) = u(z) - \psi\left(x, \frac{z}{h} \eta(x)\right). \quad (32)$$

Thus ζ and ϕ are Λ_0 -periodic and even in x functions, but they are small when a perturbed flow is close to that defined by the pair (u, h) , which is nothing else than the stream solution (U, h) written in the variables (x, z) .

The next step is to describe how ϕ depends on λ and ζ which, in its turn, defines the mapping $(\lambda, \zeta) \mapsto \phi_z(x, h)$. Therefore, the latter function is considered in the final equations which follow from (29) and (30) as the image of an operator applied to ζ . We begin with the weak setting of problem (25)–(27), and the corresponding integral identity is as follows:

$$\int_{\mathcal{D}} [\psi_x v_x + \psi_y v_y - (1 + \lambda)^2 \omega(\psi) v] dx dy = 0. \quad (33)$$

It is obtained from equation (25) by integration by parts and is valid for all $v \in W_{\text{loc}}^{1,2}(\mathcal{D})$ vanishing on $\partial\mathcal{D}$; by $W_{\text{loc}}^{1,2}(\mathcal{D})$ we denote the space each element of which belongs to $W^{1,2}(K)$ for some bounded open subset $K \subset \mathcal{D}$.

In identity (33), we change the variables (x, y) and the function ψ to (x, z) and ϕ , respectively. Moreover, we take into account the Λ_0 -periodicity and evenness of η , and the fact that ϕ is also periodic and even function of x . This allows us to integrate over $(-\Lambda_0/2, \Lambda_0/2)$ instead of \mathbb{R} (it is convenient to take $(-\Lambda_0/2, \Lambda_0/2)$ as an interval of periodicity), and so we arrive at the identity

$$\int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h \left\{ \left[\phi_x - \frac{z\eta_x}{\eta} (u_z + \phi_z) \right] \left[v_x - \frac{z\eta_x}{\eta} v_z \right] + \left(\frac{h}{\eta} \right)^2 (u_z + \phi_z) v_z - (1 + \lambda)^2 \omega(u + \phi) v \right\} \frac{\eta dx dz}{h} = 0. \quad (34)$$

It is valid for all $v \in W^{1,2}(S)$ that are vanishing on ∂S and Λ_0 -periodic in x . Furthermore, we have that

$$\int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h u_z v_z \frac{h dx dz}{\eta} = \int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h \omega(u) v \frac{h dx dz}{\eta},$$

because u in (34) is the chosen stream solution. Using this equality and the fact that $\eta_x = \zeta_x$, we transform (34) into the following form:

$$\begin{aligned} \int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h \left\{ \left[\phi_x - \frac{z\zeta_x}{\eta} (u_z + \phi_z) \right] \left[v_x - \frac{z\zeta_x}{\eta} v_z \right] + \left(\frac{h}{\eta} \right)^2 \phi_z v_z + \left[\left(\frac{h}{\eta} \right)^2 \omega(u) - (1 + \lambda)^2 \omega(u + \phi) \right] v \right\} \frac{\eta dx dz}{h} = 0. \end{aligned}$$

Let us rewrite this integral identity as follows:

$$\begin{aligned} & \int_{-\frac{\Lambda_0}{2}}^{\frac{\Lambda_0}{2}} \int_0^h \left\{ \left(\phi_x - \frac{z\zeta_x}{\eta} \phi_z \right) \left(v_x - \frac{z\zeta_x}{\eta} v_z \right) + \left(\frac{h}{\eta} \right)^2 \phi_z v_z \right. \\ & \quad \left. - (1+\lambda)^2 [\omega(u+\phi) - \omega(u)] v \right\} \frac{\eta \, dx \, dz}{h} \\ &= \int_{-\frac{\Lambda_0}{2}}^{\frac{\Lambda_0}{2}} \int_0^h \left\{ \frac{z\zeta_x}{\eta} u_z \left(v_x - \frac{z\zeta_x}{\eta} v_z \right) - \frac{(1+\lambda)^3 [2h + (1+\lambda)\zeta] \zeta}{[h + (1+\lambda)\zeta]^2} \omega(u) v \right\} \frac{\eta \, dx \, dz}{h}. \quad (35) \end{aligned}$$

Here v belongs to the same class as in (34), whereas η is the shorthand for the first term in the right-hand side in the first formula (32) (note that it depends on ζ and λ).

The existence of ϕ satisfying (35) is a consequence of assumption (III) provided ζ and λ are sufficiently small. Precisely, this is formulated as the second part of the following assertion proved at the end of section 2.3.

Proposition 2.1. *Let assumption (III) hold, and let the pair (λ, ζ) belongs to a sufficiently small neighbourhood of $(0, 0)$ in $\mathbb{R} \times \Pi_{\Lambda_0}^{2,\alpha}$. If $f \in \Pi_{\Lambda_0}^{0,\alpha}(\bar{S})$, then the weak problem*

$$\begin{aligned} & \int_{-\frac{\Lambda_0}{2}}^{\frac{\Lambda_0}{2}} \int_0^h \left\{ \left[\phi_x - \frac{z\zeta_x}{\eta} \phi_z \right] \left[v_x - \frac{z\zeta_x}{\eta} v_z \right] + \left[\frac{h}{\eta} \right]^2 \phi_z v_z - (1+\lambda)^2 [\omega(u+\phi) - \omega(u)] v \right\} \frac{\eta \, dx \, dz}{h} \\ &= \int_{-\frac{\Lambda_0}{2}}^{\frac{\Lambda_0}{2}} \int_0^h f v \, dx \, dz, \quad (36) \end{aligned}$$

where v and η are the same as in (35), has the $\Pi_{\Lambda_0}^{2,\alpha}(\bar{S})$ -solution ϕ that satisfies the estimate $\|\phi\|_{\Pi_{\Lambda_0}^{2,\alpha}(\bar{S})} \leq C \|f\|_{\Pi_{\Lambda_0}^{0,\alpha}(\bar{S})}$, where the constant C is independent of f .

In particular, the mapping $(\lambda, \zeta) \mapsto \phi$ defined by (35) continuously maps a small neighbourhood of $(0, 0)$ in $\mathbb{R} \times \Pi_{\Lambda_0}^{2,\alpha}$ to the subspace of $\Pi_{\Lambda_0}^{2,\alpha}(\bar{S})$ whose elements vanish on ∂S ; again the estimate $\|\phi\|_{\Pi_{\Lambda_0}^{2,\alpha}(\bar{S})} \leq C \|\zeta\|_{\Pi_{\Lambda_0}^{2,\alpha}}$ holds with C independent of ζ .

Now we turn to deriving a single operator equation for ζ from the Bernoulli's equation (29). Applying (32), it is straightforward to get

$$\frac{\partial \psi}{\partial n}(x, \eta(x)) = [u_z(h) + \phi_z(x, h)] \frac{h}{\eta(x)} - \frac{\phi_x(x, h) \eta_x(x)}{\sqrt{1 + \eta_x^2(x)}}.$$

Furthermore, $\phi_x(x, h)$ vanishes identically, and so (29) takes the form:

$$[u_z(h) + \phi_z(x, h)] \frac{h}{\eta(x)} - (1+\lambda) [3r - 2(1+\lambda) \eta(x)]^{1/2} = 0, \quad x \in \mathbb{R}.$$

As above η stands for the first term in the right-hand side of the first formula (32). Finally, we put $\kappa = u_z(h)$ [it is equal to $(3r - 2h)^{1/2}$ when $u_z(h) > 0$], thus obtaining after simple algebra the following form of Bernoulli's equation:

$$\phi_z(x, h) - \left[1 + \frac{1+\lambda}{h} \zeta(x) \right] [\kappa^2 - 2(1+\lambda) \zeta(x)]^{1/2} + \kappa = 0. \quad (37)$$

Here $x \in (-\Lambda_0/2, \Lambda_0/2)$, the values of h and κ are given [of course, both of them are functions of r because h depends on r through the root s_* of equation (14)], whereas ζ and λ are unknown. Moreover, ϕ depends on ζ and λ through the weak formulation (35), and so $\phi_z(x, h)$ is the image of a nonlinear operator mapping (ζ, λ) into this function on $(-\Lambda_0/2, \Lambda_0/2)$.

We recall that equation (37) is obtained under the assumption that $U'(h) > 0$. Supposing that $U'(h) < 0$, we have $\kappa = u_z(h) = -(3r - 2h)^{1/2}$, and so Bernoulli's equation takes the following form

$$\phi_z(x, h) + \left[1 + \frac{1 + \lambda}{h} \zeta(x)\right] [\kappa^2 - 2(1 + \lambda) \zeta(x)]^{1/2} + \kappa = 0, \quad (38)$$

instead of (37); here again $x \in (-\Lambda_0/2, \Lambda_0/2)$.

2.3 Linearization of equations (37) and (38)

Here we linearize equations (37) and (38), thus arriving to a single linear equation whose role is twofold. First, it serves for deriving the dispersion equation (18) (see section 2.4); second, it is used when one applies the Crandall–Rabinowitz theorem (see Theorem 4.1 in section 4) to equations (37) and (38).

Proposition 2.2. *The linearized form of equations (37) and (38) is as follows:*

$$\varphi_z^{(1)}(x, h) - \left(\frac{\kappa}{h} - \frac{1}{\kappa}\right) \zeta(x) = 0, \quad x \in (-\Lambda_0/2, \Lambda_0/2). \quad (39)$$

Here $\varphi^{(1)}$ — the leading term in the expansion of ϕ — is an even, Λ_0 -periodic function of x that satisfies the integral identity:

$$\int_{-\frac{\Lambda_0}{2}}^{\frac{\Lambda_0}{2}} \int_0^h \left[\varphi_x^{(1)} v_x + \varphi_z^{(1)} v_z - \omega'(u) \varphi^{(1)} v \right] dx dz = \frac{1}{h} \int_{-\frac{\Lambda_0}{2}}^{\frac{\Lambda_0}{2}} \int_0^h [z u_z \zeta_x v_x + 2 \omega(u) \zeta v] dx dz, \quad (40)$$

where v belongs to the same class as in (34).

Proof. The fact that $\varphi^{(1)}$ is even and Λ_0 -periodic in x is a direct consequence of (39) because ζ has these properties.

Now we turn to deriving (39). In (37) and (38), the only linear term with respect to ϕ is the derivative $\phi_z(x, h)$. Besides, it is straightforward to check that (40) is obtained by linearization of the integral identity (35) defining ϕ . Thus, $\phi_z(x, h)$ must be changed to $\varphi_z^{(1)}(x, h)$ in the linearized equation.

Expanding the second term in (37) and (38), we see that κ cancels [one has to take into account that $\sqrt{\kappa^2} = -\kappa$ in (38)], whereas the first- and second-order terms containing λ , ζ and ζ_x are

$$-\left(\frac{\kappa}{h} - \frac{1}{\kappa}\right) \zeta(x) \quad \text{and} \quad -\left(\frac{\kappa}{h} - \frac{1}{\kappa}\right) \lambda \zeta(x) + \left(\frac{1}{h\kappa} + \frac{1}{2\kappa^4}\right) [\zeta(x)]^2, \quad (41)$$

respectively. Now we change the second term in (37) and (38) to the first of these expressions; the first term in these equations we change to $\varphi_z^{(1)}(x, h)$. Thus we arrive at the linearized equation (39).

Remark 2.2. Using assumption (III) and the well-known results on the regularity of solutions to elliptic boundary value problems (see, for example, [1]), one obtains that the problem

$$\varphi_{xx} + \varphi_{zz} + \omega'(u) \varphi = f \quad \text{in } S, \quad \varphi(x, 0) = 0, \quad \varphi(x, h) = 0,$$

has a unique solution that belongs to $\Pi_{\Lambda_0}^{2,\alpha}(\bar{S})$ provided $f \in \Pi_{\Lambda_0}^{0,\alpha}(\bar{S})$. Moreover, the estimate

$$\|\varphi\|_{\Pi_{\Lambda_0}^{2,\alpha}(\bar{S})} \leq C \|f\|_{\Pi_{\Lambda_0}^{0,\alpha}(\bar{S})} \quad (42)$$

holds with the constant C independent of f .

Of course, this assertion is true for the following particular case:

$$\varphi_{xx}^{(1)} + \varphi_{zz}^{(1)} + \omega'(u) \varphi^{(1)} = h^{-1} [z u_z \zeta_{xx} - 2\omega(u) \zeta] \text{ in } S, \quad \varphi^{(1)}(x, 0) = 0, \quad \varphi^{(1)}(x, h) = 0, \quad (43)$$

where ζ is an even, Λ_0 -periodic $C^{2,\alpha}$ -function. However, the norm in the right-hand side of inequality (42) must be changed to $\|\zeta\|_{\Pi_{\Lambda_0}^{2,\alpha}}$ in this case. Note that the latter problem is the differential form of (40).

Proof of Proposition 2.1. Let us consider the nonlinear and linear operators defined by the left-hand sides of the integral identities (36) and (40), respectively. Since the pair (λ, ζ) belongs to a sufficiently small neighbourhood of $(0, 0)$ in $\mathbb{R} \times \Pi_{\Lambda_0}^{2,\alpha}$, it is easy to see that the nonlinear operator is a small perturbation of the linear one. According to assumption (III), the linear operator is invertible (see Remark 2.2). Combining this fact and the contraction mapping theorem, one obtains the first assertion of proposition. Moreover, the estimate of $\|\phi\|_{\Pi_{\Lambda_0}^{2,\alpha}(\bar{S})}$ is a consequence of inequality (42).

Furthermore, we use integration by parts in order to write the right-hand side of (35) in the same form as the right-hand side of (36). Then the second assertion of proposition follows by applying the first assertion to the weak problem (35).

2.4 Derivation of the dispersion equation (18)

In order to obtain the dispersion equation for waves with vorticity we apply the same procedure as in the case of irrotational waves (see [49], pp. 466 and 478, and [69], pp. 448–449); that is, we use the linearized problem. In the present framework, the original problem is equivalently reduced either to equation (37) or to (38) (which of these equations arises depends on the vorticity distribution). Both of them give the same linear relation (39) between ζ and $\varphi^{(1)}$ (the latter is the leading term in the expansion of ϕ). Along with (39), both these functions appear in the boundary value problem (43). Hence for deriving equation (18) it is sufficient to exclude them from (39) and (43).

Since ζ and $\varphi^{(1)}$ are even functions of x , we seek them in the form similar to the principal term in formula (20), that is, $\hat{\zeta}(\tau) \cos \tau x$ and $\hat{\varphi}^{(1)}(\tau, z) \cos \tau x$, respectively. Then we get that the following relation holds for $\tau \in \mathbb{R}$:

$$\hat{\varphi}_z^{(1)}(\tau, h) = \left(\frac{\kappa}{h} - \frac{1}{\kappa} \right) \hat{\zeta}(\tau), \quad \text{where } \hat{\varphi}^{(1)}(\tau, z) \text{ satisfies the problem} \quad (44)$$

$$-\hat{\varphi}_{zz}^{(1)} + [\tau^2 - \omega'(u)] \hat{\varphi}^{(1)} = h^{-1} [\tau^2 z u_z + 2\omega(u)] \hat{\zeta}, \quad z \in (0, h); \quad \hat{\varphi}^{(1)}(\tau, 0) = 0, \quad \hat{\varphi}^{(1)}(\tau, h) = 0. \quad (45)$$

According to assumption (III), this problem has a unique solution. Let $G(z, t; \tau)$ denote Green's function of problem (45), then we have that

$$\hat{\varphi}^{(1)}(\tau, z) = \frac{\hat{\zeta}(\tau)}{h} \int_0^h G(z, t; \tau) [\tau^2 t u_t + 2\omega(u(t))] dt.$$

Let us show that

$$G_z(z, t; \tau)|_{z=h} = -\gamma(t, \tau), \quad (46)$$

where γ is a solution of problem (19). Denoting by L the differential operator in the left-hand side of equation in problem (19) [the same operator appears in the equation of problem (45)], we write the following representation:

$$\gamma(t, \tau) = \int_0^h [\gamma(z, \tau) (L G)(z, t; \tau) - G(z, t; \tau) (L \gamma)(z, \tau)] dz.$$

It is clear that the integral is equal to

$$[\gamma_z(z, \tau) G(z, t; \tau) - G_z(z, t; \tau) \gamma(z, \tau)]_{z=0}^{z=h} = -G_z(z, t; \tau)|_{z=h}.$$

The latter equality is a consequence of the boundary conditions (19) and (45). Combining the last two formulae, we arrive at (46).

Now we differentiate the formula expressing $\hat{\varphi}^{(1)}(\tau, z)$ with respect to z and use (46), thus obtaining

$$\hat{\varphi}_z^{(1)}(\tau, h) = -\sigma_0(\tau) \hat{\zeta}(\tau), \quad \text{where} \quad \sigma_0(\tau) = h^{-1} \int_0^h \gamma(z, \tau) [\tau^2 z u_z + 2\omega(u(z))] dz. \quad (47)$$

Comparing this and (44), we get the following form of the dispersion equation:

$$\sigma_0(\tau) + \frac{\kappa}{h} - \frac{1}{\kappa} = 0.$$

In order to obtain (18), it remains to show that the left-hand side is equal to

$$\sigma(\tau) = \kappa \gamma'(h, \tau) - \kappa^{-1} + \omega(1). \quad (48)$$

Problem (19) and the problem for u allow us to calculate the integral that defines $\sigma_0(\tau)$. Since $\omega(u) = -u_{zz}$ on the interval $(0, h)$, we have that $u_z \omega'(u) = -u_{zzz}$ there. Besides, the first relation (19) gives that $\tau^2 \gamma = \gamma_{zz} + \gamma \omega'(u)$. From these facts we get that

$$\sigma_0(\tau) = \frac{1}{h} \int_0^h \{z [\gamma_{zz} + \gamma \omega'(u)] u_z - 2\gamma u_{zz}\} dz = \frac{1}{h} \int_0^h \{z [\gamma_{zz} u_z - \gamma u_{zzz}] - 2\gamma u_{zz}\} dz.$$

Integrating by parts and applying the boundary conditions for γ , we obtain

$$\int_0^h z \gamma u_{zzz} dz = -h \omega(1) - \int_0^h (z \gamma)_z u_{zz} dz.$$

Here we also used the relations $u_{zz}(h) = -\omega(u(h)) = -\omega(1)$, which follow from the definition of u . This gives

$$\sigma_0(\tau) = \omega(1) + \frac{1}{h} \int_0^h [z (\gamma_{zz} u_z + \gamma_z u_{zz}) - \gamma u_{zz}] dz = \omega(1) + \frac{1}{h} \int_0^h [z (\gamma_z u_z)_z - \gamma u_{zz}] dz.$$

Integrating by parts in the last integral, we see that it is equal to the integrated terms only; furthermore, taking into account the boundary conditions for γ and the definition of κ , we arrive at formula (48).

3 On roots of the dispersion equation (18)

In this section, we consider the dispersion equation (18) corresponding to some stream solution. First, we prove sufficient conditions which guarantee that (18) has at least one positive root, and so the first part of assumption (II) is fulfilled. Second, we show that (18) has no roots in the case when this equation corresponds to a stream solution defined by a *supercritical* value of s , that is, s is greater than s_c and satisfies the equation $\mathcal{R}_0^{(+)}(s) = r$, where r is any number greater than r_c . The latter fact is analogous to what is well known for zero vorticity, namely, that only solitary waves exist in the supercritical case (see, for example, [2] and [48]).

3.1 Sufficient conditions for solubility of equation (18)

Let assumption (I) hold, and so there exists s_* that solves equation (14) for some \mathcal{R} , and let (U, h) be the stream solution corresponding to s_* . (Here we use the simplified notation introduced above, namely, (U, h) stands for the stream solution, whereas \mathcal{R} denotes the left-hand side of equation (14) whose root s_* defines this solution.) We begin with two auxiliary assertions concerning the function σ defined by (U, h) . The first of them describes the behaviour of $\sigma(\tau)$ at infinity, and the second assertion gives the asymptotics of $\gamma(Y, \tau)$ near an eigenvalue (under the assumption that it exists) of the operator $d^2/dY^2 + \omega'(U)$ with the Dirichlet boundary conditions. (We recall that only a finite number of such eigenvalues can exist.) Then we prove the proposition about solubility of equation (18).

Lemma 3.1. *The following asymptotic formula holds*

$$\sigma(\tau) = |\tau| U'(h) + O(1) \quad \text{as } \tau \rightarrow +\infty. \quad (49)$$

Proof. In order to prove this formula we consider the solution $\gamma(Y, \tau)$ of the boundary value problem (19) and investigate its asymptotic behaviour as $\tau \rightarrow +\infty$. Note that $\gamma(Y, \tau)$ is a smooth function of both variables for large values of τ , because the operator $d^2/dY^2 + \omega'(U)$ has only a finite number of the Dirichlet eigenvalues. Let us write

$$\gamma(Y, \tau) = \frac{\sinh Y\tau}{\sinh h\tau} + \tilde{\gamma}(Y, \tau),$$

and find the asymptotics of $\tilde{\gamma}(Y, \tau)$ for large $|\tau|$. Since this function satisfies the following problem:

$$-\tilde{\gamma}'' + [\tau^2 - \omega'(U)]\tilde{\gamma} = \omega'(U) \frac{\sinh Y\tau}{\sinh h\tau} \quad \text{for } Y \in (0, h), \quad \tilde{\gamma}(0) = 0, \quad \tilde{\gamma}(h) = 0,$$

we have

$$\int_0^h (\tilde{\gamma}')^2 dY + \tau^2 \int_0^h \tilde{\gamma}^2 dY - \int_0^h \omega'(U(Y)) \tilde{\gamma}^2 dY = \int_0^h \omega'(U(Y)) \frac{\sinh Y\tau}{\sinh h\tau} \tilde{\gamma} dY.$$

This is obtained by multiplying the equation by $\tilde{\gamma}$, integrating over $(0, h)$ and using the boundary conditions after integration by parts.

Let C_ω bounds ω' from above, then the last equality yields that

$$\int_0^h \left[(\tilde{\gamma}')^2 + \frac{\tau^2}{2} \tilde{\gamma}^2 \right] dY \leq \frac{\tau^2}{4} \int_0^h \tilde{\gamma}^2 dY + \frac{4C_\omega^2}{\tau^2 \sinh^2 h\tau} \int_0^h \sinh^2 Y\tau dY, \quad (50)$$

provided τ is sufficiently large, and so the following estimate holds

$$\tilde{\gamma}(Y, \tau) = O(\tau^{-2}) \quad \text{as } |\tau| \rightarrow \infty. \quad (51)$$

Indeed, inequality (50) gives

$$\int_0^h \left[(\tilde{\gamma}')^2 + \frac{\tau^2}{4} \tilde{\gamma}^2 \right] dY \leq \frac{2C_\omega^2}{\tau^3},$$

and so

$$\max_{Y \in [0, h]} \tilde{\gamma}^2(Y, \tau) \leq 2 \left(\int_0^h (\tilde{\gamma}')^2 dY \right)^{1/2} \left(\int_0^h \tilde{\gamma}^2 dY \right)^{1/2} \leq \frac{8C_\omega^2}{\tau^4}$$

for large values of $|\tau|$. The last inequality immediately yields (51).

Furthermore, we have

$$\frac{\sinh Y\tau}{\sinh h\tau} = e^{|\tau|(Y-h)} + O\left(e^{-|\tau|h}\right) \quad \text{as } |\tau| \rightarrow \infty,$$

which combined with (51) gives that

$$\gamma(Y, \tau) = e^{|\tau|(Y-h)} + O(\tau^{-2}) \quad \text{as } |\tau| \rightarrow \infty.$$

Substituting this into the formula for σ obtained in section 2.4 [its principal integral term is given by the second relation (47)], we get

$$\sigma(\tau) = \frac{\tau^2}{h} \int_0^h e^{|\tau|(Y-h)} Y U'(Y) dY + O(1) \quad \text{as } |\tau| \rightarrow \infty.$$

The asymptotic formula (49) is a direct consequence of this representation.

Now we turn to the assertion about the behaviour of $\sigma(\tau)$ when τ^2 is close to a Dirichlet eigenvalue of the operator $d^2/dY^2 + \omega'(U)$. Prior to formulating the result, we notice that every such eigenvalue is simple (provided there exists any).

Lemma 3.2. *Let the operator $d^2/dY^2 + \omega'(U)$ considered on $(0, h)$ have τ_*^2 as a non-zero Dirichlet eigenvalue. Then the following asymptotic formula holds*

$$\sigma(\tau) = \frac{-\kappa[\gamma'_*(h)]^2}{2\tau_*(\tau - \tau_*)} + O(1) \quad \text{as } \tau \rightarrow \tau_*. \quad (52)$$

Here $\gamma_*(Y)$ is the corresponding eigenfunction normalized in $L^2(0, h)$, and $\gamma'_*(h) \neq 0$.

Proof. The constant $\gamma'_*(h)$ is not equal to zero because otherwise the function γ_* has the zero Cauchy data at $Y = h$, and so vanishes identically on $(0, h)$ which is impossible.

It is clear that τ^2 is not an eigenvalue of $d^2/dY^2 + \omega'(U)$ provided τ is sufficiently close to τ_* . Let $\gamma(Y, \tau)$ be a solution of problem (19) for such a value of τ . If we show that

$$\gamma(Y, \tau) = \frac{-\gamma'_*(h) \gamma_*(Y)}{2\tau_*(\tau - \tau_*)} + v(Y, \tau) \quad \text{as } \tau \rightarrow \tau_*, \quad (53)$$

then (52) follows from this formula in view of the definition of σ [see (18) and (19)]. Notice that the first term in the right-hand side is invariant under changing of the sign of γ_* and the remainder $v(Y, \tau)$ is a smooth function of both variables.

In order to prove the asymptotic formula (53) we use the following representation

$$\gamma(Y, \tau) = \frac{C \gamma_*(Y)}{\tau^2 - \tau_*^2} + v(Y, \tau)$$

for τ close to τ_* ; here C is a non-zero constant. In order to find C we substitute the right-hand side into (19) and get that v must satisfy the following problem:

$$-v'' + [\tau^2 - \omega'(U)]v = -C\gamma_*, \quad v(0) = 0, \quad v(h) = 1. \quad (54)$$

Therefore, $v(Y, \tau)$ exists only if $C = -\gamma'_*(h)$. Indeed, multiplying (54) by γ_* , integrating the result over $(0, h)$ and then putting $\tau = \tau_*$, we obtain that

$$C = \int_0^h \gamma_* \{v'' - [\tau_*^2 - \omega'(U)]v\} dY = \int_0^h (v''\gamma_* - \gamma_*''v) dY,$$

because γ_* is a normalized Dirichlet eigenfunction of $d^2/dY^2 + \omega'(U)$ corresponding to τ_*^2 . Now, we integrate by parts in the last integral and take into account the boundary conditions for v and γ_* . This immediately gives that $C = -\gamma'_*(h)$. Then formula (53) follows from the representation for γ with the function $v(Y, \tau)$ found from problem (54). It is clear that this function exists and is smooth.

Now we are in a position to prove the proposition about the solubility of the dispersion equation (18).

Proposition 3.3. *Let assumption (I) hold, and so there exists s_* that solves equation (14) for some \mathcal{R} . Let (U, h) be the stream solution corresponding to s_* . Then the following assertions are true for the dispersion equation with σ defined by (U, h) .*

(i) *If the operator $d^2/dY^2 + \omega'(U)$ considered on $(0, h)$ has no Dirichlet eigenvalues, then*

$$\sigma(0) = -\frac{2}{3\kappa} \left[\frac{d\mathcal{R}}{ds}(s_*) \middle/ \frac{dh}{ds}(s_*) \right]. \quad (55)$$

Moreover, if

$$\frac{d\mathcal{R}}{ds}(s_*) \middle/ \frac{dh}{ds}(s_*) > 0, \quad (56)$$

then equation (18) has at least one positive root.

(ii) *Let the operator $d^2/dY^2 + \omega'(U)$ considered on $(0, h)$ have no zero Dirichlet eigenvalue. If this operator has exactly k positive Dirichlet eigenvalues, then (18) has at least k positive roots. Moreover, if inequality (56) holds, then (18) has at least $k + 1$ positive roots.*

Proof. (i) Since the operator $d^2/dY^2 + \omega'(U)$ has no Dirichlet eigenvalues, σ is a smooth function. Therefore, it is sufficient to evaluate $\sigma(0)$ in order to prove that (56) guarantees the existence of a root. First, we consider the case when $\kappa = U'(h) > 0$, and show that condition (56) is equivalent to the following inequality:

$$\sigma(0) < 0. \quad (57)$$

Then assertion (i) is an immediate consequence of the last two inequalities and the asymptotic formula (49), according to which $\sigma(\tau) \rightarrow +\infty$ as $\tau \rightarrow +\infty$

For proving (55) and (57) we note that

$$\sigma(0) = \kappa \gamma'(h, 0) - \kappa^{-1} + \omega(1) \quad (58)$$

by the definition of σ . On the one hand, $\gamma(Y, 0)$ satisfies the boundary value problem (19) with $\tau = 0$ and $(U(Y; s_*), h(s_*))$. On the other hand, the Cauchy problem for a general stream function $U(Y; s)$ is as follows:

$$U'' + \omega(U) = 0, \quad U(0) = 0, \quad U'(0) = s.$$

Differentiating the first two relations with respect to s , we get

$$\dot{U}'' + \omega'(U) \dot{U} = 0 \quad \text{for } U \in (0, h), \quad \dot{U}(0) = 0.$$

(Here and below \dot{U} denotes the derivative with respect to the parameter s .) Comparing these relations and the problem for $\gamma(Y, 0)$, we see that

$$\gamma(Y, 0) = \frac{\dot{U}(Y; s_*)}{\dot{U}(h(s_*); s_*)}, \quad (59)$$

where the denominator does not vanish because $\gamma(Y, 0)$ is well-defined by the boundary value problem (19) with $\tau = 0$. Now we differentiate (59) and substitute the result into (58), where κ is changed to $U'(h(s_*))$ in the first term. Then we get that

$$\sigma(0) = \frac{U'(h(s_*)) \dot{U}'(h(s_*); s_*)}{\dot{U}(h(s_*); s_*)} - \frac{1}{\kappa} + \omega(1) = \frac{[\{U'(h(s))\}^2]_s}{2 \dot{U}(h(s); s)} \Big|_{s=s_*} - \frac{1}{\kappa} + \omega(1).$$

Applying relation (8), we obtain

$$\sigma(0) = \frac{s_* - \omega(h(s_*)) \dot{U}(h(s_*); s_*)}{\dot{U}(h(s_*); s_*)} - \frac{1}{\kappa} + \omega(1) = -\frac{1}{\kappa} \left[1 + \frac{s_*}{h(s_*)} \right].$$

Here $U'(h(s_*))$ is changed back to κ ; moreover the equality $h(s_*) = 1$ and the formula

$$U'(h(s_*)) \dot{h}(s_*) + \dot{U}(h(s_*); s_*) = 0 \quad (60)$$

are used. The last equality arises when one differentiates the condition $U(h(s); s) = 1$ (it holds for every stream solution) with respect to s and puts $s = s_*$. Finally, differentiating \mathcal{R} with respect to s (see formula (14), where \mathcal{R} the is defined), we get that

$$\sigma(0) = -\frac{1}{\kappa} \left[1 + \frac{s_*}{h(s_*)} \right] = -\frac{2}{3\kappa} \left[\frac{d\mathcal{R}}{ds}(s_*) \right] / \left[\frac{dh}{ds}(s_*) \right]. \quad (61)$$

In view of this formula, inequality (57) is a consequence of (56) when $\kappa = U'(h) > 0$. This proves (57) and completes the proof of the proposition when the last inequality is assumed to hold.

It remains to consider the case when $\kappa = U'(h) < 0$. According to the asymptotic formula (49), this condition yields that $\sigma(\tau) \rightarrow -\infty$ as $\tau \rightarrow +\infty$. Hence, if we show that inequality (56) implies that

$$\sigma(0) > 0, \quad (62)$$

then we immediately obtain the required assertion. Using formula (61) (it is independent of the sign of κ) and the inequality $\kappa < 0$, we see that (62) is again equivalent to (56). Thus the proof of assertion (i) is complete.

(ii) First, let exactly one Dirichlet eigenvalue $\tau_*^2 > 0$ of $d^2/dY^2 + \omega'(U)$ exist. Then formula (53) and the definition of σ yield that $\sigma(\tau) \rightarrow \mp\infty$ as $\tau \rightarrow \tau_* + 0$ provided $\pm\kappa > 0$. On the other hand, $\sigma(\tau) \rightarrow \pm\infty$ as $\tau \rightarrow +\infty$ by virtue of the asymptotic formula (49). By the assumption there is no eigenvalue other than τ_*^2 . Hence $\sigma(\tau)$ is smooth for $\tau \in (\tau_*, +\infty)$ and it tends to opposite infinities at the ends of this interval. Then at least one root of equation (18) exists on $(\tau_*, +\infty)$.

Since zero is not an eigenvalue, $\sigma(0)$ is defined. Therefore, it is easy to modify the proof of (i) so that it will combine inequality (56) and the limit of $\sigma(\tau)$ as $\tau \rightarrow \tau_* - 0$ instead of the limit as $\tau \rightarrow +\infty$. On this way, one obtains that one more root of (18) exists on the interval $(0, \tau_*)$.

Now, let us assume that there are exactly two Dirichlet eigenvalues $[\tau_*^{(1)}]^2$ and $[\tau_*^{(2)}]^2 > [\tau_*^{(1)}]^2$. Let $\gamma_*^{(1)}$ and $\gamma_*^{(2)}$, respectively, denote the corresponding eigenfunctions. Then it is easy to see that

$$\sigma(\tau) = \frac{-\kappa \left[(\gamma_*^{(j)})'(h) \right]^2}{2\tau_*^{(j)} (\tau - \tau_*^{(j)})} + O(1) \quad \text{as } \tau \rightarrow \tau_*^{(j)}, \quad j = 1, 2,$$

which is similar to (52). Thus, $\sigma(\tau)$ tends to opposite infinities (their signs depend on the sign of κ) as τ goes to $\tau_*^{(1)} + 0$ and $\tau_*^{(2)} - 0$. Hence there exists at least one root of equation (18) on the interval $(\tau_*^{(1)}, \tau_*^{(2)})$. As in the case of a single eigenvalue, at least one more root belongs to $(\tau_*^{(2)}, +\infty)$, and so the total number of roots is not less than two. The case of k eigenvalues should be treated in the same way. The proof is complete.

3.2 The dispersion equation for unidirectional flows

First, a stream solution (U, h) describes a unidirectional flow, that is, $U'(Y) > 0$ for all $Y \in [0, h]$, only when this solution depends on $s > s_0$ which is a root of equation (14) with $\mathcal{R}_0^{(+)}(s)$ (see [50], sections 5.2 and 5.3). Moreover, we have (see [50], section 5.1):

$$\pm \frac{d\mathcal{R}_0^{(+)}}{ds}(s) > 0 \quad \text{provided } \pm(s - s_c) > 0. \quad (63)$$

Second, the corresponding operator $d^2/dY^2 + \omega'(U)$ considered on $(0, h)$ has the empty Dirichlet spectrum. Hence the solution $\gamma(Y; \tau)$ of problem (19) is defined for all $\tau \in \mathbb{R}$ and is a smooth function of both variables. The same is true for σ that stands in the dispersion equation because

$$\sigma(\tau) = \sigma(0) + \kappa v'(h; \tau), \quad \text{where } v(Y; \tau) = \gamma(Y; \tau) - \gamma(Y; 0) = \gamma(Y; \tau) - \frac{\dot{U}(Y)}{\dot{U}(h)}, \quad (64)$$

where the last equality is a consequence of formula (59). Of course, v depends also on the parameter $s > s_0$ and, for the sake of brevity, this dependence is indicated by the dot standing for the differentiation with respect to s . Note that this function does not vanish identically on $(0, h)$.

Lemma 3.4. *Let $s > s_0$ be a root of equation (14) with $\mathcal{R}_0^{(+)}$ in the left-hand side. If the function v is defined by formula (64), then $v(Y; \tau) \leq 0$ for all $Y \in [0, h]$ and $v'(h; \tau) \geq 0$; both inequalities hold for all $\tau \in \mathbb{R}$.*

Proof. The definitions of U and γ and formula (59) yield that v satisfies the following problem:

$$v'' + \omega'(U)v = \tau^2 \gamma \quad \text{on } (0, h), \quad v(0) = v(h) = 0.$$

Note that $\gamma \geq 0$ on $[0, h]$, which is a consequence of formula (3.9) in [50]. Putting $w = v/U'$ and taking into account the equality $U''' = -U'\omega'(U)$, we rewrite the previous boundary value problem as follows:

$$w'' + 2\frac{U''}{U'}w' = \frac{\tau^2 \gamma}{U'} \quad \text{on } (0, h), \quad w(0) = w(h) = 0.$$

The right-hand side of the equation is non-negative because $U'(Y) > 0$ for all $Y \in [0, h]$. Therefore, the maximum principle is applicable (see [53], Chapter 1, Corollary to Theorem 4). Hence $w \leq 0$ on $(0, h)$ and the same is true for v . Combining this and the boundary conditions for v , we get that $v'(h; \tau) \geq 0$ for all $\tau \in \mathbb{R}$. The proof is complete.

Now we are in a position to prove the result concerning the dispersion equation (18) for unidirectional flows.

Proposition 3.5. *Let $s > s_0$ be a root of the equation $\mathcal{R}_0^{(+)}(s) = r$ for some $r > r_c$. If the function σ is defined by the stream solution (U, h) corresponding to this s , then the following assertions hold.*

- (i) *For any $s \in (s_0, s_c)$ equation (18) has a positive solution.*
- (ii) *Equation (18) has no positive solutions for $s > s_c$.*

Proof. Assertion (i) (it is included for the sake of completeness) is an immediate consequence of Proposition 3.3, (i). Indeed, inequality (56) follows from (63) with the lower sign and

$$\frac{dh_0^{(+)}}{ds}(s) < 0. \quad (65)$$

The last inequality holds for all $s > s_0$ according to formula (4.1) in [50].

In order to prove (ii), let us take an arbitrary $s > s_c$ and assume that the corresponding equation (18) has a positive solution τ_0 . Writing formula (48) in the form

$$\sigma(\tau_0) = -\omega(1) + \frac{\kappa}{h(s)} - \kappa \gamma'(h(s), \tau_0) = -\omega(1) + \frac{\kappa}{h(s)} - \kappa \gamma'(h(s), 0) - \kappa v'(h(s), \tau_0),$$

we get that

$$\sigma(\tau_0) \leq -\omega(1) + \frac{\kappa}{h(s)} - \kappa \gamma'(h(s), 0) \quad (66)$$

because $\kappa = U'(h(s); s) > 0$ and Lemma 3.4 gives that $v'(h(s); \tau_0) \geq 0$. Using formula (59), we transform the last term in (66) as follows:

$$-\kappa \gamma'(h, 0) = -U'(h) \frac{\dot{U}'(h)}{\dot{U}(h)} = -\frac{[(U'(h))^2]_s}{2\dot{U}(h)} = -\frac{s - \omega(U(h))}{\dot{U}(h)}.$$

Here the last equality is a consequence of equation (8). Substituting this into (66) and taking into account that $U(h) = 1$, we get that

$$\sigma(\tau_0) \leq \frac{\kappa}{h} - \frac{s}{\dot{U}(h)} = \frac{\kappa}{h} + \frac{s}{\kappa \dot{h}},$$

which (60) is also used. Subtracting equation (18) with $\tau = \tau_0$ from this inequality, we obtain that $1 + (s/\dot{h}) \geq 0$, but this cannot be true. Indeed, it is shown in the proof of Proposition 3.3 that this inequality is equivalent to

$$\frac{d\mathcal{R}_0^{(+)}(s)}{ds} \bigg/ \frac{dh_0^{(+)}(s)}{ds} \geq 0.$$

However the denominator of the last inequality is positive by (63), where the upper sign must be taken, whereas the numerator is negative by (65). Thus, the obtained contradiction proves assertion (ii).

4 Proof of Main Theorem

Our proof of Main Theorem is based on the following theorem which deals with bifurcation from a simple eigenvalue.

Theorem 4.1. *Let I be an open interval of \mathbb{R} such that $0 \in I$, and let \mathcal{X}, \mathcal{Y} be Banach spaces. If a continuous map $\mathcal{F} : I \times \mathcal{X} \mapsto \mathcal{Y}$ has the following properties:*

- (i) *the equality $\mathcal{F}(\lambda, 0) = 0$ holds for all $\lambda \in I$,*
- (ii) *the operators $\mathcal{F}_\lambda, \mathcal{F}_\zeta$ and $\mathcal{F}_{\lambda\zeta}$ exist in a neighbourhood of $(0, 0)$ and are continuous there,*
- (iii) *\mathcal{F}_ζ is a Fredholm operator, the null-space of $\mathcal{F}_\zeta(0, 0)$ is one-dimensional,*
- (iv) *if the null-space of $\mathcal{F}_\zeta(0, 0)$ is generated by $\zeta^{(0)}$, then $\mathcal{F}_{\lambda\zeta}(0, 0)\zeta^{(0)}$ does not belong to the range of $\mathcal{F}_\zeta(0, 0)$.*

Then a sufficiently small $\varepsilon > 0$ exists and a continuous curve

$$\{(\lambda(t), \zeta(t)) : |t| < \varepsilon\} \subset I \times \mathcal{X},$$

bifurcates from $(0, 0)$. Moreover, for pairs belonging to this curve the following properties hold:

$$\zeta(t) = t\zeta^{(0)} + o(t) \quad \text{when } 0 < |t| < \varepsilon,$$

and

$$\{(\lambda, \zeta) \in V : \zeta \neq 0 \text{ and } \mathcal{F}(\lambda, \zeta) = 0\} = \{(\lambda(t), \zeta(t)) : 0 < |t| < \varepsilon\},$$

where $V \subset I \times \mathcal{X}$ is a certain neighbourhood of $(0, 0)$.

This result is proved by Crandall and Rabinowitz (see Theorem 1.7 in [21]).

4.1 Application of Theorem 4.1 to equations (37) and (38)

In order to prove the first assertion of Main Theorem, we apply Theorem 4.1 to equations (37) and (38). (It is sufficient to consider only the first of them.)

Let us denote the left-hand side of equation (37) by $F(\lambda, \zeta)$. This is an operator mapping $\mathbb{R} \times \Pi_{\Lambda_0}^{2,\alpha} \mapsto \Pi_{\Lambda_0}^{1,\alpha}$. Indeed, the derivatives of ϕ in the expression for F depend on ζ and λ through the weak formulation (35). According to Remark 2.3, assumption (III) implies that $\phi \in \Pi_{0,\Lambda_0}^{2,\alpha}(\bar{S})$ provided ζ is sufficiently small. Here $\Pi_{0,\Lambda_0}^{2,\alpha}(\bar{S})$ denotes the subspace of $\Pi_{\Lambda_0}^{2,\alpha}(\bar{S})$ whose elements vanish on ∂S , that is, $\Pi_{0,\Lambda_0}^{2,\alpha}(\bar{S})$ consists of $C^{k,\alpha}$ -functions on \bar{S} that are Λ_0 -periodic and even in

x , and vanish for $z = 0$ and $z = h$. Thus, conditions (i)–(iv) of Theorem 4.1 must be verified for the equation $F(\lambda, \zeta) = 0$, in which case $\mathcal{X} = \Pi_{\Lambda_0}^{2,\alpha}$ and $\mathcal{Y} = \Pi_{\Lambda_0}^{1,\alpha}$.

Condition (i) of Theorem 4.1 follows from assumption (III) because

$$F(\lambda, 0) = \phi_z(x, h) \quad \text{for } \lambda \in I, \quad x \in (-\Lambda_0/2, \Lambda_0/2), \quad (67)$$

and we have that $\phi_z(x, h) = 0$ when $\zeta = 0$. In order to prove this fact, we substitute these two equalities into the integral identity (34), thus getting

$$\int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h \{ \phi_x v_x + (1 + \lambda)^2 \phi_z v_z + (1 + \lambda)^2 [\omega(u) - \omega(u + \phi)] v \} \, dx \, dy = 0,$$

where v belongs to the same class as in (34). Since $\omega \in C^{2,\alpha}(\mathbb{R})$, the integrand can be written as follows:

$$\phi_x v_x + \phi_z v_z - \omega'(u) \phi v + (2\lambda + \lambda^2) [\phi_z v_z - \omega'(u) \phi v] + (1 + \lambda)^2 v O(\phi^2).$$

Let \mathcal{L} denote the operator of the Λ_0 -periodic boundary value problem (43) (note that \mathcal{L} corresponds to the first three terms in the last expression). Then assumption (III) implies that

$$\mathcal{L} : \Pi_{0,\Lambda_0}^{2,\alpha}(\bar{S}) \mapsto \Pi_{0,\Lambda_0}^{0,\alpha}(\bar{S}) \quad \text{is an isomorphism}$$

(see, for example, [1]). Therefore, we have that

$$\|\phi\|_{\Pi_{0,\Lambda_0}^{2,\alpha}(\bar{S})} \leq C \|\phi\|_{\Pi_{0,\Lambda_0}^{2,\alpha}(\bar{S})}^2, \quad \text{where } C = \text{const independent of } \phi.$$

Hence ϕ is identically equal to zero provided $\|\phi\|$ is small. From this fact and (67) one gets that condition (i) is fulfilled for F .

Condition (ii) concerns the existence and continuity of the operators F_λ , F_ζ and $F_{\lambda\zeta}$ in $\Pi_{\Lambda_0}^{2,\alpha}$. According to Remark 2.3, assumption (III) implies that the mapping $(\lambda, \zeta) \mapsto \phi$ [it is defined by the weak formulation (35)] is continuous from a neighbourhood of $(0, 0)$ in $\mathbb{R} \times \Pi_{\Lambda_0}^{2,\alpha}$ to $\Pi_{0,\Lambda_0}^{2,\alpha}(\bar{S})$. Moreover, this mapping has continuous derivatives of the first and second orders.

Condition (iii). The fact that F_ζ is a Fredholm operator follows from (37). Furthermore, Proposition 2.2 yields that

$$[F_\zeta(0, 0) \varsigma](x) = (\mathcal{K}^{(1)} \varsigma)(x) - \left(\frac{\kappa}{h} - \frac{1}{\kappa} \right) \varsigma(x), \quad x \in (-\Lambda_0/2, \Lambda_0/2), \quad (68)$$

where $(\mathcal{K}^{(1)} \varsigma)(x)$ is defined as follows. Let $\Phi^{(1)}(x, z; \varsigma)$ be Λ_0 -periodic in x and satisfy the integral identity (40) with $\zeta = \varsigma$, then

$$(\mathcal{K}^{(1)} \varsigma)(x) = \Phi_z^{(1)}(x, h; \varsigma),$$

and so $\mathcal{K}^{(1)}$ is a Fredholm operator from $\Pi_{\Lambda_0}^{2,\alpha}$ to $\Pi_{\Lambda_0}^{1,\alpha}$.

Finally, conditions (a) and (b) in assumption (II) guarantee that the null-space of $F_\zeta(0, 0)$ is one-dimensional. Indeed, it is straightforward to check that this space is generated by

$$\zeta^{(0)}(x) = \cos \frac{2\pi x}{\Lambda_0}.$$

Condition (iv). In order to obtain an expression for the operator $F_{\lambda\zeta}(0, 0)$ let us consider $\varphi^{(2)}$, the second-order term in the expansion of ϕ . It follows from (35) that $\varphi^{(2)}$ satisfies the following

integral identity:

$$\begin{aligned}
& \int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h \left[\varphi_x^{(2)} v_x + \varphi_z^{(2)} v_z - \omega'(u) \varphi^{(2)} v \right] dx dz \\
&= \int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h \left\{ \frac{\omega(u)}{h} \left[2\lambda\zeta + \left(2\lambda - \frac{\zeta}{h} \right) \zeta \right] v - \left(\frac{z}{h} \right)^2 u_z \zeta_x^2 v_z \right\} dx dz \\
&\quad + \int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h \left\{ \left(\lambda - \frac{\zeta}{h} \right) \left(\varphi_x^{(1)} v_x - \varphi_z^{(1)} v_z \right) + \frac{z}{h} \left(\varphi_x^{(1)} v_z + \varphi_z^{(1)} v_x \right) \zeta_x \right. \\
&\quad \left. + \left[\left(\lambda + \frac{\zeta}{h} \right) \omega'(u) \varphi^{(1)} + \frac{\omega''(u)}{2} \left(\varphi^{(1)} \right)^2 \right] v \right\} dx dz.
\end{aligned}$$

Combining the last equality, the second expression (41) and formula (68), we get that

$$[F_{\lambda\zeta}(0,0)\varsigma](x) = \left(\mathcal{K}^{(2)}\varsigma \right)(x) - \left(\frac{\kappa}{h} - \frac{1}{\kappa} \right) \varsigma(x), \quad x \in (-\Lambda_0/2, \Lambda_0/2). \quad (69)$$

Here $(\mathcal{K}^{(2)}\varsigma)(x)$ is defined in the same way as the operator $\mathcal{K}^{(1)}$. Namely, let $\Phi^{(2)}(x, z; \varsigma)$ be a Λ_0 -periodic in x solution of the problem that has the following weak formulation:

$$\begin{aligned}
& \int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h \left[\Phi_x^{(2)} v_x + \Phi_z^{(2)} v_z - \omega'(u) \Phi^{(2)} v \right] dx dz \\
&= \int_{-\Lambda_0/2}^{\Lambda_0/2} \int_0^h \left[\frac{4\omega(u)}{h} \varsigma v + \Phi_x^{(1)}(x, z; \varsigma) v_x - \Phi_z^{(1)}(x, z; \varsigma) v_z + \omega'(u) \Phi^{(1)}(x, z; \varsigma) v \right] dx dz.
\end{aligned}$$

Here v belongs to the same space as in (34) and $\Phi^{(1)}(\cdot; \varsigma)$ is defined after formula (68). Then we have that

$$\left(\mathcal{K}^{(2)}\varsigma \right)(x) = \Phi_z^{(2)}(x, h; \varsigma). \quad (70)$$

Now we are in a position to check condition (iv). Since F_ζ is a Fredholm operator, it is sufficient to show that

$$\left\langle F_{\lambda\zeta}(0,0)\zeta^{(0)}, \zeta^{(0)} \right\rangle \neq 0, \quad (71)$$

where $\langle \cdot, \cdot \rangle$ denotes the corresponding bilinear form. In fact, (71) is a consequence of condition (a) in assumption (II) as the following assertion demonstrates.

Proposition 4.2. *In formula (71), the left-hand side is equal to $\pi \sigma'(\tau_0)$.*

Proof. It follows from (69) and (70) that

$$\left\langle F_{\lambda\zeta}(0,0)\zeta^{(0)}, \zeta^{(0)} \right\rangle = \left\langle \Phi_z^{(2)}(x, h; \zeta^{(0)}), \zeta^{(0)} \right\rangle - \frac{\Lambda_0}{2} \left(\frac{\kappa}{h} - \frac{1}{\kappa} \right),$$

because

$$\left\langle \zeta^{(0)}, \zeta^{(0)} \right\rangle = \int_{-\Lambda_0/2}^{\Lambda_0/2} \cos^2 \frac{2\pi x}{\Lambda_0} dx = \frac{\Lambda_0}{2}.$$

Seeking $\Phi_z^{(2)}(x, h; \zeta^{(0)})$, it is natural to use the same ansatz for both $\Phi^{(1)}$ and $\Phi^{(2)}$, namely:

$$\Phi^{(k)}(x, z; \zeta^{(0)}) = p^{(k)}(z) \cos \frac{2\pi x}{\Lambda_0}, \quad k = 1, 2.$$

Then we get that

$$\left\langle \left[(F_{\lambda\zeta}(0,0))\zeta^{(0)} \right], \zeta^{(0)} \right\rangle = \frac{\Lambda_0}{2} \left[p_z^{(2)}(h) - \left(\frac{\kappa}{h} - \frac{1}{\kappa} \right) \right].$$

where $p^{(2)}$ satisfies the boundary value problem:

$$-p_{zz}^{(2)} + [\tau_0^2 - \omega'(u)] p^{(2)} = \frac{4}{h} \omega(u) + p_{zz}^{(1)} + [\tau_0^2 + \omega'(u)] p^{(1)}, \quad p^{(2)}(0) = p^{(2)}(h) = 0.$$

Furthermore, the problem for $p^{(1)}$ is as follows:

$$-p_{zz}^{(1)} + [\tau_0^2 - \omega'(u)] p^{(1)} = h^{-1} [\tau_0^2 z u_z + 2\omega(u)], \quad p^{(1)}(0) = 0, \quad p^{(1)}(h) = 0, \quad (72)$$

which, in particular, implies that

$$p_z^{(1)}(h) = -\sigma_0(\tau_0) \quad (73)$$

[see the derivation of formula (47)].

Putting $p^{(2)} = 2p^{(1)} + q$ and taking into account the last equality, we see that

$$\langle F_{\lambda\zeta}(0, 0) \zeta^{(0)}, \zeta^{(0)} \rangle = \frac{\Lambda_0}{2} \left[q_z(h) + \left(\frac{\kappa}{h} - \frac{1}{\kappa} \right) \right], \quad (74)$$

where [cf. formula (47)]

$$q_z(h) = - \int_0^h \gamma(z, \tau) \left\{ p_{zz}^{(1)} + [\tau_0^2 + \omega'(u)] p^{(1)} - \frac{2}{h} \tau_0^2 z u_z \right\} dz,$$

because q satisfies the following problem:

$$-q_{zz} + [\tau_0^2 - \omega'(u)] q = p_{zz}^{(1)} + [\tau_0^2 + \omega'(u)] p^{(1)} - \frac{2}{h} \tau_0^2 z u_z, \quad q(0) = q(h) = 0.$$

Splitting the last integral into the sum of two terms and integrating by parts twice in the first one, we obtain

$$\begin{aligned} q_z(h) &= -p_z^{(1)}(h) - \int_0^h [\gamma_{zz}(z, \tau_0) + \omega'(u(z)) \gamma(z, \tau_0)] p^{(1)}(z) dz \\ &\quad - \tau_0^2 \int_0^h \gamma(z, \tau_0) \left[p^{(1)}(z) - \frac{2}{h} z u_z(z) \right] dz. \end{aligned}$$

In view of equation for γ [see (19)], this reduces to

$$q_z(h) = -p_z^{(1)}(h) - 2\tau_0^2 \int_0^h \gamma(z, \tau_0) \left[p^{(1)}(z) - h^{-1} z u_z(z) \right] dz,$$

which combined with formulae (73), (74) and the definition of τ_0 leads to

$$\langle F_{\lambda\zeta}(0, 0) \zeta^{(0)}, \zeta^{(0)} \rangle = -\Lambda_0 \tau_0^2 \int_0^h \gamma(z, \tau_0) \left[p^{(1)}(z) - h^{-1} z u_z(z) \right] dz. \quad (75)$$

Let us show that the last integral is equal to $-\sigma'_0(\tau_0)/(2\tau_0)$, which completes the proof in view of the relation between τ_0 and Λ_0 , and the equality $-\sigma'_0(\tau) = \sigma'(\tau)$ [see formula (48)]. Differentiating formula (47), we get

$$\sigma'_0(\tau_0) = h^{-1} \int_0^h \{ 2\gamma(z, \tau_0) \tau_0 z u_z(z) + \gamma'(z, \tau_0) [\tau_0^2 z u_z(z) + 2\omega(u(z))] \} dz,$$

where γ' stands for $\partial\gamma/\partial\tau$. Let us also differentiate relations (19) with respect to τ , thus obtaining

$$-\gamma'_{zz} + [\tau^2 - \omega'(u)] \gamma' = -2\tau\gamma, \quad \gamma'(0) = 0, \quad \gamma'(h) = 0. \quad (76)$$

This problem is similar to (72), and it allows us to express γ' in terms of γ (cf. section 2.4):

$$\gamma'(z, \tau) = -2\tau \int_0^h G(z, z'; \tau) \gamma(z', \tau) dz',$$

where $G(z, z'; \tau)$ is Green's function of problem (76). Using this representation for calculating the second term in braces in the formula for $\sigma'_0(\tau_0)$, we get that this term is equal to

$$\frac{2\tau_0}{h} \int_0^h [\tau_0^2 z u_z(z) + 2\omega(u(z))] dz \int_0^h G(z, z'; \tau) \gamma(z', \tau_0) dz'.$$

After changing the order of integration, we notice that the inner integral gives $h p^{(1)}(z')$. Thus, the last double integral reduces to

$$2\tau_0 \int_0^h p^{(1)}(z') \gamma(z', \tau_0) dz'.$$

Substituting this into the formula for $\sigma'(\tau_0)$, we arrive at

$$\sigma'_0(\tau_0) = 2\tau_0 \int_0^h \gamma(z, \tau_0) [p^{(1)}(z) - h^{-1} z u_z(z)] dz.$$

Comparing this and (75), we obtain the proposition's assertion.

Since conditions (i)–(iv) of Theorem 4.1 are fulfilled for equations (37) and (38), assertion (i) of Main Theorem is proved.

4.2 Proof of Main Theorem: assertion (ii)

It is noticed in Remark 2.3 that there exists a solution of the nonlinear problem whose weak formulation is given by the integral identity (35). Moreover, the map $(\lambda(t), \zeta(x, t)) \mapsto \phi(x, z, t)$ defined by (35) is continuous from a neighbourhood of $(0, 0)$ in $\mathbb{R} \times \Pi_{\Lambda_0}^{2, \alpha}$ to the space $\Pi_{0, \Lambda_0}^{2, \alpha}(\bar{S})$, and the following estimate mentioned in Remark 2.3 holds:

$$\|\phi\|_{\Pi_{\Lambda_0}^{2, \alpha}(\bar{S})} \leq C \|\zeta\|_{\Pi_{\Lambda_0}^{2, \alpha}}.$$

Now we change variables using formula (24), where $\Lambda = \Lambda_0[1 + \lambda(t)]$, and (31) and (32). This gives the function

$$\Psi(X, Y, t) = U \left(Y \frac{h}{\xi(X, t)} \right) + \phi \left(\frac{X \Lambda_0}{\Lambda}, Y \frac{h}{\xi(X, t)}, t \right) \in \Pi_{\Lambda}^{2, \alpha}(\bar{\mathcal{D}}). \quad (77)$$

In view of considerations presented in sections 2 and 4.1, Ψ solves problem (1)–(3), whereas the pair (Ψ, ξ) satisfies condition (4).

Comparing formulae (77) and (22), we see that the second term in the right-hand side of (77) is equal to Ψ_* in (22). Combining this fact, the last inequality and assertion (i) of Main Theorem, we get that the second term in the right-hand side of (22) is $O(t)$. This completes the proof of Main Theorem.

5 Examples

In this section, we consider two examples of vorticity distributions (the corresponding flows of constant depth were studied in our paper [50]), for which the dispersion equation is investigated along with condition (56). This allows us to draw conclusions about the existence of Stokes waves perturbing a flow of constant depth with vorticity. In the first example, the vorticity is equal to a positive constant, while the second example deals with the linear vorticity having a positive coefficient and vanishing at zero.

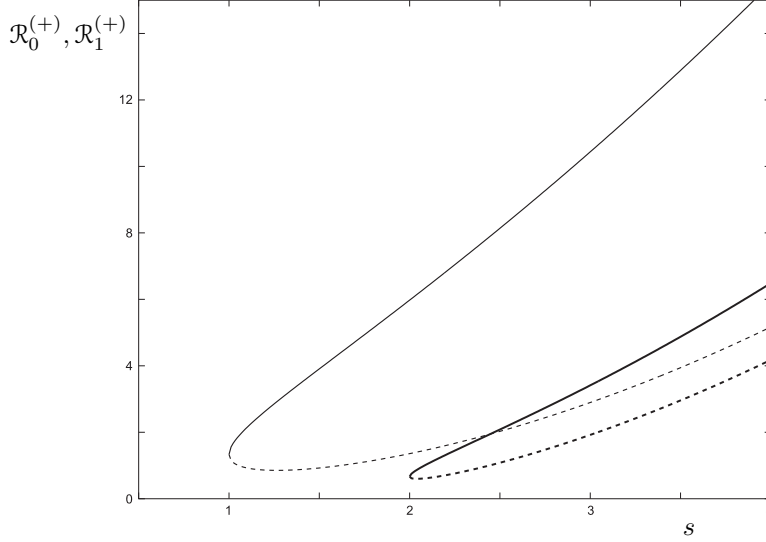


Figure 1: For the constant vorticity $\omega = b > 0$, the functions $\mathcal{R}_0^{(+)}$ (dashed lines) and $\mathcal{R}_1^{(+)}$ (bold lines) are plotted for $b = 1/2$ (left) and $b = 2$ (right).

5.1 Constant positive vorticity

Let $\omega = b$ be a positive constant, then it immediately follows (see [50] section 6.1) that $s_0 = \sqrt{2b}$, and

$$h_0^{(+)}(s) = \frac{s - \sqrt{s^2 - 2b}}{b}, \quad h_1^{(+)}(s) = \frac{s + \sqrt{s^2 - 2b}}{b} \quad (78)$$

are the only non-vanishing depth functions defined for $s \geq s_0$ and such that $h_0^{(+)}(s) \leq h_1^{(+)}(s)$ with the equality holding only for $s = s_0$ when both values are equal to $h_0 = \sqrt{2/b}$. Hence the left-hand sides in the Bernoulli's equation (14) are as follows (see Fig. 1, where these functions are plotted for $b = 1/2$ and $b = 2$; reproduced from [50], p. 391):

$$\mathcal{R}_{\frac{1}{2} \mp \frac{1}{2}}^{(+)}(s) = \frac{1}{3} \left[s^2 - 2b + 2 \frac{s \mp \sqrt{s^2 - 2b}}{b} \right]. \quad (79)$$

These functions have the following properties: $\mathcal{R}_0^{(+)}(s) < \mathcal{R}_1^{(+)}(s)$ for $s > s_0$, while

$$r_0 = \mathcal{R}_0^{(+)}(s_0) = \mathcal{R}_1^{(+)}(s_0) = \frac{2}{3} \sqrt{\frac{2}{b}} < +\infty, \quad (80)$$

and

$$\mathcal{R}_{\frac{1}{2} \mp \frac{1}{2}}^{(+)}(s) = \frac{s^2}{3} + \frac{2}{3b}(s \mp s) + O(1) \quad \text{as } s \rightarrow +\infty. \quad (81)$$

The distance between $\mathcal{R}_0^{(+)}(s)$ and $\mathcal{R}_1^{(+)}(s)$ increases with s due to the second term in the right-hand side. [This term is lost in formula (6.2), [50], analogous to (81).] Furthermore, we have that

$$\frac{d\mathcal{R}_{\frac{1}{2} \mp \frac{1}{2}}^{(+)}}{ds}(s) \rightarrow \mp \infty \quad \text{as } s \rightarrow s_0 + 0 \quad \text{and} \quad \frac{d\mathcal{R}_1^{(+)}}{ds}(s) > 0 \quad \text{for } s > s_0. \quad (82)$$

The first of these relations shows that the graphs of $\mathcal{R}_0^{(+)}$ and $\mathcal{R}_1^{(+)}$ have the common vertical tangent at the point (s_0, r_0) , while the latter inequality implies that the function $\mathcal{R}_1^{(+)}$ increases

strictly monotonically, and so its graph of lies strictly above the level r_0 . Finally, the single minimum $r_c < r_0$ of $\mathcal{R}_0^{(+)}$ is attained at $s_c > s_0$ satisfying the following equation:

$$(s_c b)^2 - 2b^3 = \left(\frac{s_c b}{1 + s_c b} \right)^2. \quad (83)$$

It is clear that the first component of the stream solution is

$$U(Y; s) = -\frac{b}{2}Y^2 + sY. \quad (84)$$

It solves the corresponding Cauchy problem [see (6) for the Cauchy data]. Since there are two options for the second component [they are given by formulae (78)], two steady flows of constant depth exist for every $r > r_c$. It is shown in [50], p. 392, that if the depth is equal to $h_0^{(+)}(s)$ and $s \in (s_0, s_c)$ is obtained from equation (14) with $\mathcal{R}_0^{(+)}$ [it will be clear from what follows why $s > s_c$ is not considered], then the flow is unidirectional because $U'(Y; s) > 0$ for all $Y \in [0, h_0^{(+)}(s)]$. On the contrary, if the depth is equal to $h_1^{(+)}(s)$ [in this case, $s > s_0$ satisfies equation (14) with $\mathcal{R}_1^{(+)}$], then there is a near-surface counter-current. Indeed, it is easy to calculate that

$$U'(h_1^{(+)}(s); s) = -\sqrt{s^2 - 2b} < 0 \quad \text{and} \quad U'(Y; s) < 0 \text{ for } Y > s/b$$

It is clear that the latter value is less than $h_1^{(+)}(s)$.

In order to write down the dispersion equation we find the following solution of problem (19):

$$\gamma(Y, \tau) = \frac{\sinh \tau Y}{\sinh \tau h}, \quad Y \in [0, h(s)].$$

Here and below $h = h(s)$ is either $h_0^{(+)}(s)$ or $h_1^{(+)}(s)$ [see (78)], and s is a root of the corresponding equation (14); the common form of both these equations is as follows:

$$s^2 + 2h(s) = 3r + 2b.$$

We substitute $\kappa = -(bh - s)$ and γ into (18) and obtain

$$\tau(bh - s) \coth \tau h - b - (bh - s)^{-1} = 0. \quad (85)$$

This dispersion equation coincides with (17) when $b = 0$. Indeed, we have that $s = h = h_0^{(+)}$ in view of the first formula (78) [the second formula (78) is meaningless for $b = 0$].

It is easy to see that the derivative of the left-hand side in (85) has a definite sign that depends on the sign of $bh - s$, and so if this equation has a root, then it is simple. Instead of demonstrating directly that (85) is soluble for both $h_0^{(+)}(s)$ and $h_1^{(+)}(s)$ [see (78)], we apply Proposition 3.3 for this purpose.

First, let $r > r_0$, then the second relation (82) implies that the function $\mathcal{R}_1^{(+)}$ increases monotonically. Thus, the inequality

$$\frac{dh_1^{(+)}}{ds}(s) = \frac{1}{b} \left(1 + \frac{s}{\sqrt{s^2 - 2b}} \right) > 0$$

guarantees that condition (56) is fulfilled, and so Proposition 3.3 yields that (85) with $h = h_1^{(+)}(s)$ has a positive root. In this case, the left-hand side in (85) is a monotonically increasing function of τ because $bh - s = \sqrt{s^2 - 2b} > 0$. Hence the dispersion equation (85) with $h = h_1^{(+)}(s)$ has only one positive root.

Let us turn to the case when $h = h_0^{(+)}(s)$ and $s \in (s_0, s_c)$. Then we have that $r \in (r_c, r_0)$ and

$$\frac{dh_0^{(+)}}{ds}(s) = \frac{1}{b} \left(1 - \frac{s}{\sqrt{s^2 - 2b}} \right) < 0 \quad \text{for all } s > s_0.$$

Moreover, in view of (83) the following inequality holds

$$\frac{d\mathcal{R}_0^{(+)}}{ds}(s) = \frac{2}{3} \left[s + \frac{1}{b} \left(1 - \frac{s}{\sqrt{s^2 - 2b}} \right) \right] < 0 \quad \text{for } s \in (s_0, s_c).$$

Hence Proposition 3.3 yields that the dispersion equation (85) with $h = h_0^{(+)}(s)$ has a positive root. Again, there is only one root because the left-hand side in (85) is a monotonically decreasing function of τ , since $bh - s = -\sqrt{s^2 - 2b} < 0$ in the present case.

On the basis of the above considerations we formulate the following.

Proposition 5.1. *Let $\omega = b$, where b is a positive constant, then the following assertions hold.*

(i) *For every $r > r_0$ [see (80) for r_0] the equation $\mathcal{R}_1^{(+)}(s) = r$ [see (79) for $\mathcal{R}_1^{(+)}$] has the unique root $s_*^{(1)} > s_0$. In the flow of the constant depth $h = h_1^{(+)}(s_*^{(1)})$ [see the second formula (78)], the velocity field is described by the stream function (84). The dispersion equation (85) with $s = s_*^{(1)}$ and $h = h_1^{(+)}(s_*^{(1)})$ has only one positive simple root.*

(ii) *For every $r \in (r_c, r_0)$ [relations (83) and (79) that give s_c and $\mathcal{R}_0^{(+)}$, respectively, show how to find r_c] the equation $\mathcal{R}_0^{(+)}(s) = r$ has the unique root $s_*^{(0)} \in (s_0, s_c)$. In the flow of the constant depth $h = h_0^{(+)}(s_*^{(0)})$ [see the first formula (78)], the velocity field is described by the stream function (84). The dispersion equation (85) with $s = s_*^{(0)}$ and $h = h_0^{(+)}(s_*^{(0)})$ has only one positive simple root.*

Combining this proposition and Main Theorem, we arrive at the following conclusion.

Corollary 5.2. *Let $\omega = b$, where b is a positive constant. Then for every $r > r_c$, $r \neq r_0$, there exists a family of Stokes waves with wavelengths belonging to a neighbourhood of Λ_0 such that $2\pi/\Lambda_0$ is the only positive root of equation (85) depending on r as described in Proposition 5.1. These small-amplitude waves perturb each of the flows whose constant depths are given by formulae (78), where s depends on r through the corresponding equation (14).*

Remark 5.3. When $\omega = b$ is a positive constant, the family of Stokes waves, existing for every $r > r_0$ by Corollary 5.2, perturbs a flow of constant depth that has a near-surface counter-current. If $r \in (r_c, r_0)$, then the corresponding family of Stokes waves perturbs a unidirectional flow of constant depth. In the case when $\omega = b < 0$, our approach gives the results similar to those formulated in Proposition 5.1 and Corollary 5.2. The corresponding flows of constant depth (they have near-bottom counter-currents), are described in [50], section 6.2.

Remark 5.4. Further details about small-amplitude Stokes waves with constant vorticity can be found in [68] and [20]. In the paper [68], Wahlén paid much attention to the behaviour of streamlines for such a flow. In particular, he investigated in detail the streamlines that form a ‘critical layer’ which separates two layers with the opposite directions of flow. Moreover, within this layer, all streamlines are closed, that is, it consists of the so-called cat’s-eye vortices (see figures 1 and 3 in [68], where $\omega = b < 0$). Besides, a full description of particle paths is given in [68]. The approach developed for the same problem by Constantin and Varvaruca [20] is applicable to a wider set of free surface profiles, in particular, it includes overhanging ones.

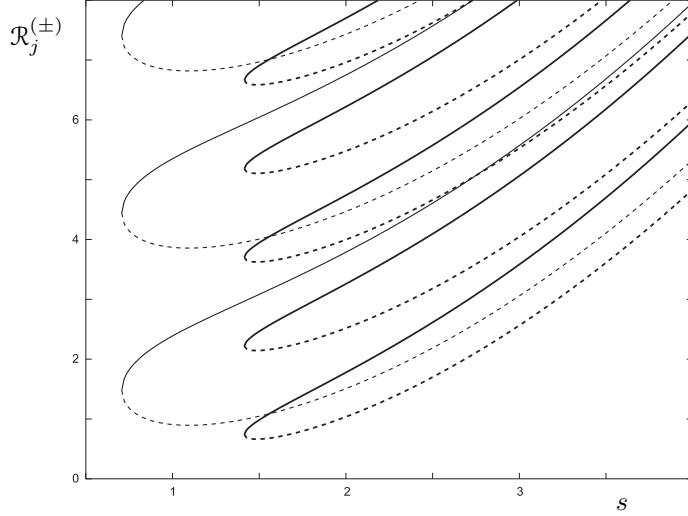


Figure 2: For the vorticity $\omega = b\tau$, $b > 0$, the interlacing pairs of functions $\mathcal{R}_{2k}^{(+)}$ (dashed lines) and $\mathcal{R}_{2k+1}^{(+)}$ (bold lines), and $\mathcal{R}_{2k}^{(-)}$ (dashed lines) and $\mathcal{R}_{2k+1}^{(-)}$ (bold lines), $k = 0, 1, \dots$, are plotted for $b = 1/2$ (tips are closer to the vertical axis) and $b = 2$.

5.2 Linear positive vorticity

Let $\omega = b\tau$, where b is a positive constant. Then $\Omega(\tau) = b\tau^2/2$, and so $s_0 = \sqrt{b}$. According to formulae (4.4), (4.5) and (4.8) in [50], the following two sequences

$$h_j^{(+)}(s) = \frac{(-1)^j}{\sqrt{b}} \arcsin \frac{\sqrt{b}}{s} + j \frac{\pi}{\sqrt{b}} \quad \text{and} \quad h_j^{(-)}(s) = h_j^{(+)}(s) + \frac{\pi}{\sqrt{b}}, \quad j = 0, 1, \dots, \quad (86)$$

give the depths of flows with horizontal free surfaces. Therefore, the left-hand-side terms in equation (14) are as follows (see Fig. 2, where the corresponding graphs are plotted for $b = 1/2$ and $b = 2$; reproduced from [50]):

$$\mathcal{R}_j^{(\pm)}(s) = \frac{1}{3} \left[s^2 - b + \frac{2(-1)^j}{\sqrt{b}} \arcsin \frac{\sqrt{b}}{s} + \frac{2\pi}{\sqrt{b}} \left(j + \frac{1}{2} \mp \frac{1}{2} \right) \right], \quad j = 0, 1, \dots \quad (87)$$

In particular, we have that

$$\mathcal{R}_0^{(+)}(s) = \frac{1}{3} \left(s^2 - b + \frac{2}{\sqrt{b}} \arcsin \frac{\sqrt{b}}{s} \right) \quad \text{and} \quad \mathcal{R}_1^{(+)}(s) = \frac{1}{3} \left(s^2 - b + \frac{2\pi}{\sqrt{b}} - \frac{2}{\sqrt{b}} \arcsin \frac{\sqrt{b}}{s} \right).$$

(In each of the two families shown in Fig. 2, the graphs corresponding to these functions are plotted as the lowest dashed and solid lines, respectively.) Hence

$$r_0 = \mathcal{R}_0^{(+)}(s_0) = \mathcal{R}_1^{(+)}(s_0) = \frac{\pi}{3\sqrt{b}},$$

whereas r_c — the only minimum of $\mathcal{R}_0^{(+)}(s)$ — is attained at

$$s_c = \sqrt{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + 1}}. \quad (88)$$

Furthermore, both relations (82) remain true in the present case together with the conclusions drawn from them about the pattern formed by the graphs of $\mathcal{R}_0^{(+)}$ and $\mathcal{R}_1^{(+)}$. Moreover, this pattern is repeated infinitely many times; indeed, formulae (87) imply that it is shifted by $2\pi j/(3\sqrt{b})$, $j = 1, 2, \dots$, along the positive r -axis (see Fig. 2).

The first component of a stream solution has the form

$$U(Y; s) = \pm \frac{s}{\sqrt{b}} \sin \sqrt{b} Y. \quad (89)$$

Indeed, this function solves the Cauchy problem that consists of the first and second relations (5). The sign in (89) coincide with that in the superscript of the corresponding $h_j^{(\pm)}(s)$, which is the second component of the same stream solution. Infinitely many options of it are given by formulae (86); however, only a finite number of them defines steady flows of constant depth for a particular value of r greater than r_c . The number depends on how many roots has the corresponding set of equations (14) for a chosen r . Note that the number of roots increases with r .

Prior to investigating which flows support Stokes waves we turn to the sequence of dispersion equations which exists in the present case. Indeed, the left-hand side of (18) involves γ that solves the following problem:

$$-\gamma'' + (\tau^2 - b)\gamma = 0, \quad \gamma(0) = 0, \quad \gamma(h) = 1, \quad (90)$$

where $h = h_j^{(\pm)}(s)$, $j = 1, 2, \dots$. Each of these values depends on a root s of equation (14) with $\mathcal{R}_j^{(\pm)}$ in the left-hand side. Applying the remark that follows problem (19), we conclude that (90) has a solution for all τ such that

$$\tau^2 \neq b - \left(\frac{\pi k}{h}\right)^2, \quad k = 1, 2, \dots \quad (91)$$

This condition is obviously fulfilled when $\tau^2 \geq b$, in which case $\gamma(Y, \tau)$ is equal to

$$\begin{aligned} & \bullet \quad Y/h \quad \text{when } \tau^2 = b; \\ & \bullet \quad \frac{\sinh \sqrt{\tau^2 - b} Y}{\sinh \sqrt{\tau^2 - b} h} \quad \text{when } \tau^2 > b, \end{aligned}$$

and so γ (and σ as well) is a continuous function of τ when $\tau^2 \geq b$.

Using formula (89) with $h = h_j^{(\pm)}(s)$ and the second of the bulleted expressions in formula (18), we get that the sequence of dispersion equations for $\tau^2 \geq b$ has the form:

$$s^2 \sqrt{\tau^2 - b} \cos^2 \left(\sqrt{b} h_j^{(\pm)}(s) \right) \coth \left(\sqrt{\tau^2 - b} h_j^{(\pm)}(s) \right) - 1 \pm bs \cos \left(\sqrt{b} h_j^{(\pm)}(s) \right) = 0, \quad j = 0, 1, \dots, \quad (92)$$

and the left-hand side must be understood as the corresponding limit for $\tau^2 = b$.

If $\tau^2 < b$ and condition (91) is fulfilled, then

$$\gamma(Y, \tau) = \frac{\sin \sqrt{b - \tau^2} Y}{\sin \sqrt{b - \tau^2} h}.$$

Substituting (89) and this expression into formula (18), we get the sequence of dispersion equations for this case:

$$s^2 \sqrt{b - \tau^2} \cos^2 \left(\sqrt{b} h_j^{(\pm)}(s) \right) \cot \left(\sqrt{b - \tau^2} h_j^{(\pm)}(s) \right) - 1 \pm bs \cos \left(\sqrt{b} h_j^{(\pm)}(s) \right) = 0, \quad j = 0, 1, \dots \quad (93)$$

The limit form of (93) as $\tau^2 \rightarrow b - 0$ coincides with that of (92) as $\tau^2 \rightarrow b + 0$. It is clear that the derivatives of the left-hand sides of (92) and (93) have definite signs on the intervals of continuity, and so if these equations have roots, then they are simple. In order to investigate equations (92) and (93) we apply Proposition 3.3.

Taking into account formula (91), we begin with flows such that $\sqrt{b}h_j^{(\pm)}(s) < \pi$. Formulae (86) show that only $h_0^{(+)}(s)$ and $h_1^{(+)}(s)$ satisfy this inequality. Moreover, Proposition 3.3 (i) guarantees that for each of these depths equations (92) and (93) have at least one positive root provided condition (56) is fulfilled for either of the following pairs:

$$\mathcal{R} = \mathcal{R}_0^{(+)}, \quad h = h_0^{(+)}; \quad \mathcal{R} = \mathcal{R}_1^{(+)}, \quad h = h_1^{(+)}.$$

(In fact, the form of the left-hand side in (92) and (93) allows the existence of only one root.) The results obtained in [50], section 6.3, yield that inequality (56) always holds for the second pair of functions, whereas for the first pair it holds only when $r \in (r_c, r_0)$. [We recall that $r_0 = \pi/3\sqrt{b}$, whereas $r_c = \mathcal{R}_0^{(+)}(s_c)$ and s_c is given by formula (88).]

In the first case, the depth is $h_0^{(+)}(s)$, where $s \in (s_0, s_c)$ is obtained from equation (14) with $\mathcal{R}_0^{(+)}$ and $r \in (r_c, r_0)$. Any such flow is unidirectional because of the upper sign in formula (89) which gives that

$$U'(Y; s) = s \cos \sqrt{b}Y > 0 \quad \text{for all } Y \in [0, h_0^{(+)}(s)].$$

On the other hand, Proposition 3.5 is applicable when $\mathcal{R} = \mathcal{R}_0^{(+)}$ and $h = h_0^{(+)}$. Therefore, if σ is defined by the stream solution corresponding to the root $s > s_c$ of $\mathcal{R}_0^{(+)}(s) = r$ with an arbitrary $r > r_c$, then the dispersion equations (92) and (93) have no positive solutions.

In the second case, the depth is $h_1^{(+)}(s)$, where $s > s_0$ is obtained from equation (14) with $\mathcal{R}_1^{(+)}$ and an arbitrary $r > r_0$. All such flows have a near-surface counter-current because

$$U'(Y; s) = s \cos \sqrt{b}Y \quad \text{changes sign only once on } (0, h_0^{(+)}(s)).$$

Combining the above considerations and Main Theorem, we formulate the following.

Proposition 5.5. *Let $\omega = b\tau$, where b is a positive constant, then the following assertions hold.*

(i) *For every $r \in (r_c, r_0)$ there exists a family of Stokes waves with wavelengths belonging to a neighbourhood of Λ_0 such that $2\pi/\Lambda_0$ is the only positive simple root of (92) and (93) in which $s \in (s_0, s_c)$ solves the equation $\mathcal{R}_0^{(+)}(s) = r$. These small-amplitude waves perturb the unidirectional flow having the constant depth $h = h_0^{(+)}(s)$ [see the first formula (86)].*

Equations (92) and (93) have no roots when $s > s_c$ is a solution of the equation $\mathcal{R}_0^{(+)}(s) = r$.

(ii) *For every $r > r_0$ there exists a family of Stokes waves with wavelengths belonging to a neighbourhood of Λ_0 such that $2\pi/\Lambda_0$ is the only positive simple root of (92) and (93) in which $s > s_0$ solves the equation $\mathcal{R}_1^{(+)}(s) = r$. These small-amplitude waves perturb the flow with a near-surface counter-current; the constant depth of this flow is equal to $h = h_1^{(+)}(s)$ [see the first formula (86)].*

If $\sqrt{b}h_j^{(\pm)}(s) \in (\pi, 2\pi)$, then formulae (86) show that only $h_0^{(-)}(s)$ and $h_1^{(-)}(s)$ satisfy this condition provided s is a root of $\mathcal{R}_0^{(-)}(s) = r$ and $\mathcal{R}_1^{(-)}(s) = r$, respectively. It is clear that the second of these equations has only one solution for any $r > r_0 + \frac{2\pi}{3\sqrt{b}}$ and the range of solutions is $(s_0, +\infty)$. Furthermore, the equation $\mathcal{R}_0^{(-)}(s) = r$ has only one solution belonging to $(s_c, +\infty)$ for any $r > r_c + \frac{2\pi}{3\sqrt{b}}$. Moreover, for every $r \in (r_c + \frac{2\pi}{3\sqrt{b}}, r_0 + \frac{2\pi}{3\sqrt{b}})$ the latter equation has another solution in (s_0, s_c) . Thus, the pair of equations $\mathcal{R}_0^{(-)}(s) = r$ and $\mathcal{R}_1^{(-)}(s) = r$ has a pair of solutions for all $r > r_c + \frac{2\pi}{3\sqrt{b}}$ except for $r = r_0 + \frac{2\pi}{3\sqrt{b}}$, when only the first of these equations has a solution.

All flows, whose constant depth is equal either to $h_0^{(-)}(s)$ or to $h_1^{(-)}(s)$, have near-bottom counter-currents. In the first case, there is no other counter-current, but the near-surface counter-current is also present in the second case.

Now we substitute $h_0^{(-)}(s)$ and $h_1^{(-)}(s)$ corresponding to the values of s obtained in the previous paragraph into equations (92) and (93) and see that Proposition 3.3 (ii) is applicable. Indeed, formula (91) yields that $\tau_*^2 = b - (\pi/h)^2$ with $h = h_k^{(-)}(s)$, $k = 0, 1$, is the single Dirichlet eigenvalue of the operator $d^2/dY^2 + \omega'(U)$ considered on $(0, h)$. It follows from the cited proposition that equations (92) and (93) with $h_k^{(-)}(s)$, $k = 0, 1$, have at least one positive root $\tau_+^{(k)} \in (\tau_*, +\infty)$. In fact, the form of the left-hand sides in (92) and (93) yields the existence of only one root there.

On the other hand, if condition (56) is fulfilled for the following pairs

$$\mathcal{R} = \mathcal{R}_0^{(-)}, \quad h = h_0^{(-)}, \quad \text{and} \quad \mathcal{R} = \mathcal{R}_1^{(-)}, \quad h = h_1^{(-)},$$

then Proposition 3.3 (i) is applicable. From the results obtained in [50], section 6.3, it follows that (56) is always fulfilled for the second pair of functions, whereas for the first pair it holds only when $s \in (s_0, s_c)$, that is, when $r \in (r_c + 2\pi/3\sqrt{b}, r_0 + 2\pi/3\sqrt{b})$ in equation (14) with $\mathcal{R}_0^{(-)}$ in the left-hand side. Then Proposition 3.3 (i) guarantees that equations (92) and (93) with $h_k^{(-)}(s)$, $k = 0, 1$, have at least one positive root $\tau_-^{(k)} \in (0, \tau_*)$. (Again, the form of the left-hand sides in (92) and (93) yields the existence of only one root there.)

Combining the above considerations and Main Theorem, we formulate the following.

Proposition 5.6. *Let $\omega = b\tau$, where b is a positive constant. Let $s > s_0$ be an arbitrary solution of $\mathcal{R}_k^{(-)}(s) = r$, $k = 0, 1$, with $r > r_c + 2\pi/3\sqrt{b}$ (there always exist two such solutions except when $r = r_0 + 2\pi/3\sqrt{b}$, in which case only one solution exists). Then the following assertions hold.*

(i) *Equations (92) and (93) with $h = h_k^{(-)}(s)$ have a positive simple root $\tau_+^{(k)} > [b - (\pi/h)^2]^{1/2}$. There exists a family of Stokes waves with wavelengths belonging to a neighbourhood of $2\pi/\tau_+^{(k)}$, and these small-amplitude waves perturb the flow of the constant depth $h_k^{(-)}(s)$. Moreover, a near-bottom counter-current is present in the flow when $k = 0, 1$, whereas for $k = 1$ a near-surface counter-current also exists.*

(ii) *If $s > s_0$ is a solution of $\mathcal{R}_1^{(-)}(s) = r$ with $r > r_0 + 2\pi/3\sqrt{b}$, or if $s \in (s_0, s_c)$ is a solution of $\mathcal{R}_0^{(-)}(s) = r$ with $r \in (r_c + 2\pi/3\sqrt{b}, r_0 + 2\pi/3\sqrt{b})$, then equations (92) and (93) with $h = h_k^{(-)}(s)$, $k = 0, 1$, have a positive simple root $\tau_-^{(k)} < [b - (\pi/h)^2]^{1/2}$. There exists a family of Stokes waves with wavelengths belonging to a neighbourhood of $2\pi/\tau_-^{(k)}$, and these small-amplitude waves perturb the flow of the constant depth $h_k^{(-)}(s)$. Again, a near-bottom counter-current is present in the flow when $k = 0, 1$, whereas for $k = 1$ a near-surface counter-current also exists.*

If $\sqrt{b}h_j^{(\pm)}(s) \in (2\pi, 3\pi)$, then formulae (86) show that only $h_2^{(+)}(s)$ and $h_3^{(+)}(s)$ satisfy this condition provided s is a root of $\mathcal{R}_2^{(+)}(s) = r$ and $\mathcal{R}_3^{(+)}(s) = r$, respectively. The same analysis shows that these equations have the same roots as in the previous case provided $2\pi/3\sqrt{b}$ is added to r that stands in the right-hand side of $\mathcal{R}_0^{(-)}(s) = r$ and $\mathcal{R}_1^{(-)}(s) = r$, respectively. But now, formula (91) yields the existence of two Dirichlet eigenvalues, namely, $\tau_{*1}^2 = b - (\pi/h)^2$ and $\tau_{*2}^2 = b - (2\pi/h)^2$ with $h = h_k^{(+)}(s)$, $k = 2, 3$, for the operator $d^2/dY^2 + \omega'(U)$ considered on $(0, h)$. Hence Proposition 5.6 can be easily reformulated for this case. In fact, the form of assertion (ii) remains the same, but there is no near-bottom counter-current and the number of counter-currents is different. Besides, assertion (i) should be modified in view that more roots of equations (92) and (93) exist. The required amendments immediately follow from Proposition 3.3 (ii), and are left to the reader.

In conclusion, we summarize what was said above about Stokes waves on flows with positive linear vorticity. First, if $r > r_c$, then every root of each equation $\mathcal{R}_j^{(\pm)}(s) = r$ [see formula (87) for

$\mathcal{R}_j^{(\pm)}(s)$], defines a stream solution from which small-amplitude Stokes waves bifurcate, unless this root is equal to s_0 or is greater than s_c and satisfies the equation $\mathcal{R}_0^{(+)}(s) = r$. Second, the number of stream solutions from which small-amplitude Stokes waves bifurcate increases as r grows. On the one hand, for larger values of r more equations $\mathcal{R}_j^{(\pm)}(s) = r$ give roots that can be used in the dispersion equations (92) and (93). On the other hand, the latter equations have more roots defining bifurcation wavelengths because larger roots of $\mathcal{R}_j^{(\pm)}(s) = r$ exist for greater values of r .

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